# AP Calc BC

# UNIT 9

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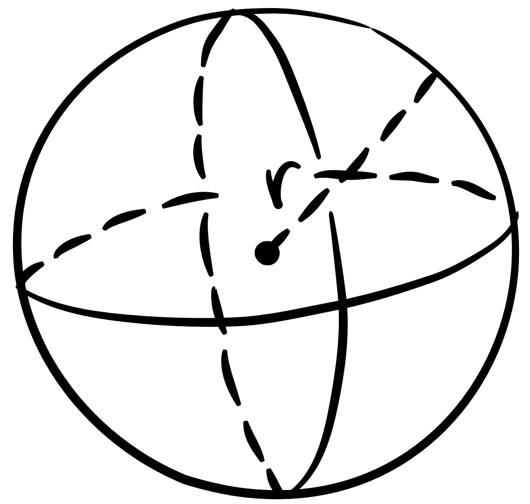
# 9.1 - Defining and Differentiating Parametric Equations

**Parametric Equation** - a type of equation that uses a third variable known as a parameter (usually t) to define the x and y coordinates.

Instead of defining y in terms of x (such as y = f(x)) or x in terms of y (such as x = g(y)), parametric equations are defined in terms of t as
(x, y) = (f(t), g(t)) with x = f(t) and y = g(t).

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.1 - Defining and Differentiating Parametric Equations

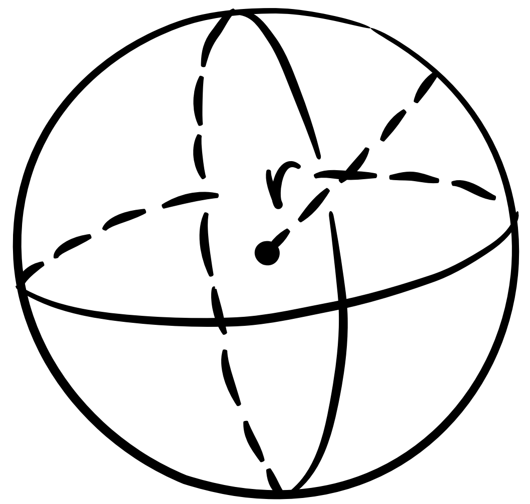
**First derivative of a parametric function:**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} \text{ for } \frac{dx}{dt} \neq 0$$

Think about cancelling out dt

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

# 9.1 - Defining and Differentiating Parametric Equations

A curve in the plane is defined parametrically by the equations $x = 2\cos(3t)$ and $y = -3\sin(2t)$.

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

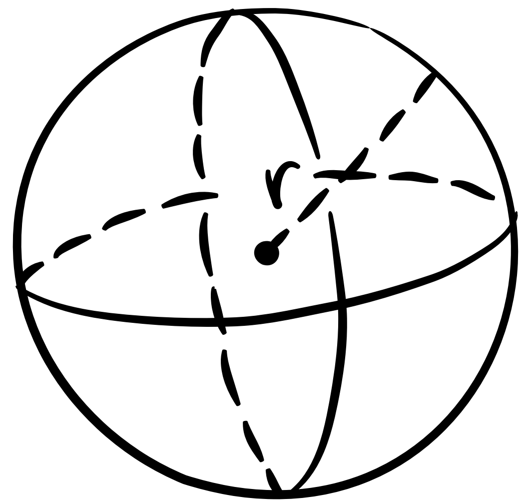
$$\frac{dx}{dt} = -6\sin(3t)$$

$$\frac{dy}{dt} = -6\cos(2t)$$

$$= \frac{-6\cos(2t)}{-6\sin(3t)} = \boxed{\frac{\cos(2t)}{\sin(3t)}}$$

$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

# 9.1 - Defining and Differentiating Parametric Equations

A curve in the plane is defined parametrically by the equations $x = \ln(4t-3)$ and $y = \dfrac{2}{t}$.

**Find the value of $\dfrac{dy}{dx}$ at $t = 3$.**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{2}{t^2}}{\frac{4}{4t-3}} = \frac{-\frac{1}{t^2}\cdot\frac{t^2}{1}}{\frac{2}{4t-3}\cdot\frac{t^2}{1}} = \frac{-1\cdot\frac{(4t-3)}{1}}{\frac{2t^2}{4t-3}\cdot\frac{(4t-3)}{1}}$$

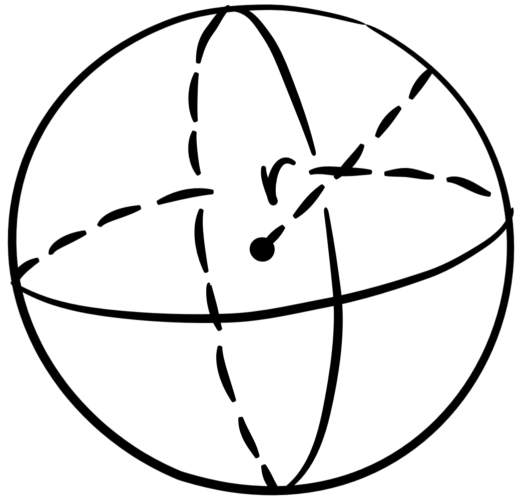
$$\frac{dx}{dt} = \frac{4}{4t-3} \qquad \frac{dy}{dt} = -\frac{2}{t^2}$$

$$= -\frac{4t-3}{2t^2}\Bigg|_{t=3}$$

$$= -\frac{4(3)-3}{2(9)} = \boxed{-\frac{1}{2}}$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

# 9.2 - Second Derivatives of Parametric Equations

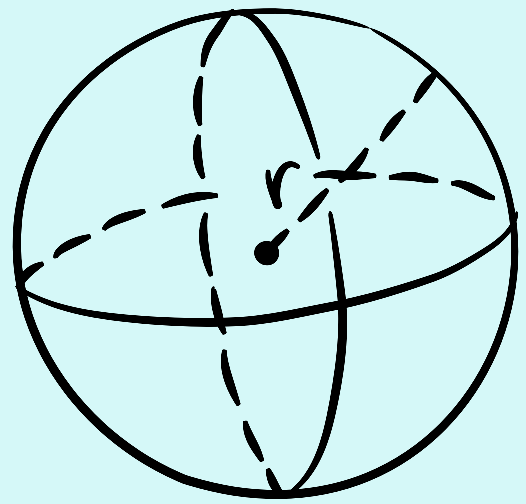
## REMEMBER TO USE THIS FORMULA

**Derivatives Of A Function In Parametric Form**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.2 - Second Derivatives of Parametric Equations

A curve is defined by the parametric equations
$x = 3^t - 1$ and $y = 9^t$.

$\frac{dx}{dt} = (\ln 3)3^t \qquad \frac{dy}{dt} = (\ln 9)9^t$

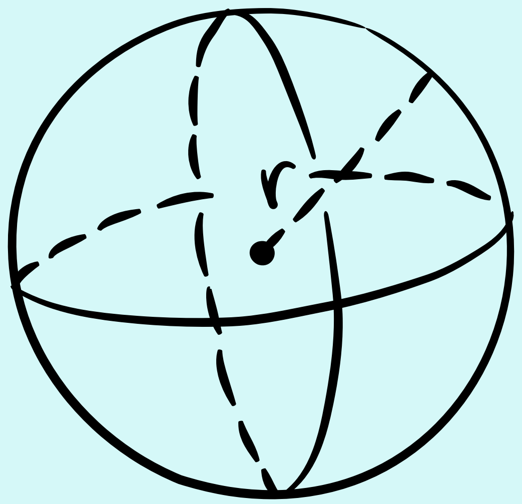
What is $\frac{d^2y}{dx^2}$ in terms of $t$ ?

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(\ln 9)9^t}{(\ln 3)3^t} = \frac{(\ln 3^2)3^{2t}}{(\ln 3)3^t} = \frac{(2\ln 3)3^t}{(\ln 3)} = 2(3^t)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2(3)^t)}{(\ln 3)3^t} = \frac{2(\cancel{\ln 3})\cancel{3^t}}{(\cancel{\ln 3})\cancel{3^t}} = \boxed{2}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

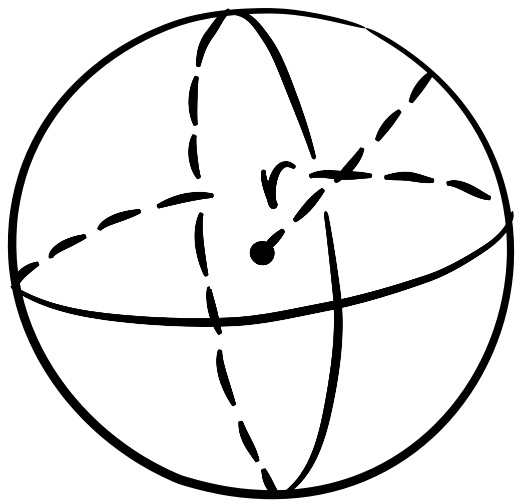
# 9.3 - Arc Length W/ Parametric Equations

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.3 - Arc Length W/ Parametric Equations

Consider the parametric curve:

$$x = \frac{-5}{t^2} \qquad \frac{dx}{dt} = \frac{10}{t^3}$$

$$y = e^{-t} \qquad \frac{dy}{dt} = -e^{-t}$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

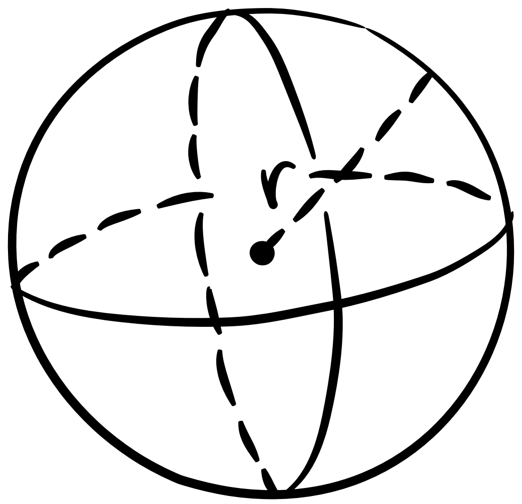
$$L = \int_{-5}^{0} \sqrt{\left(\frac{10}{t^3}\right)^2 + \left(-e^{-t}\right)^2}\,dt$$

$$= \int_{-5}^{0} \sqrt{\frac{100}{t^6} + e^{-2t}}\,dt$$

Which integral gives the arc length of the curve over the interval from $t = -5$ to $t = 0$?

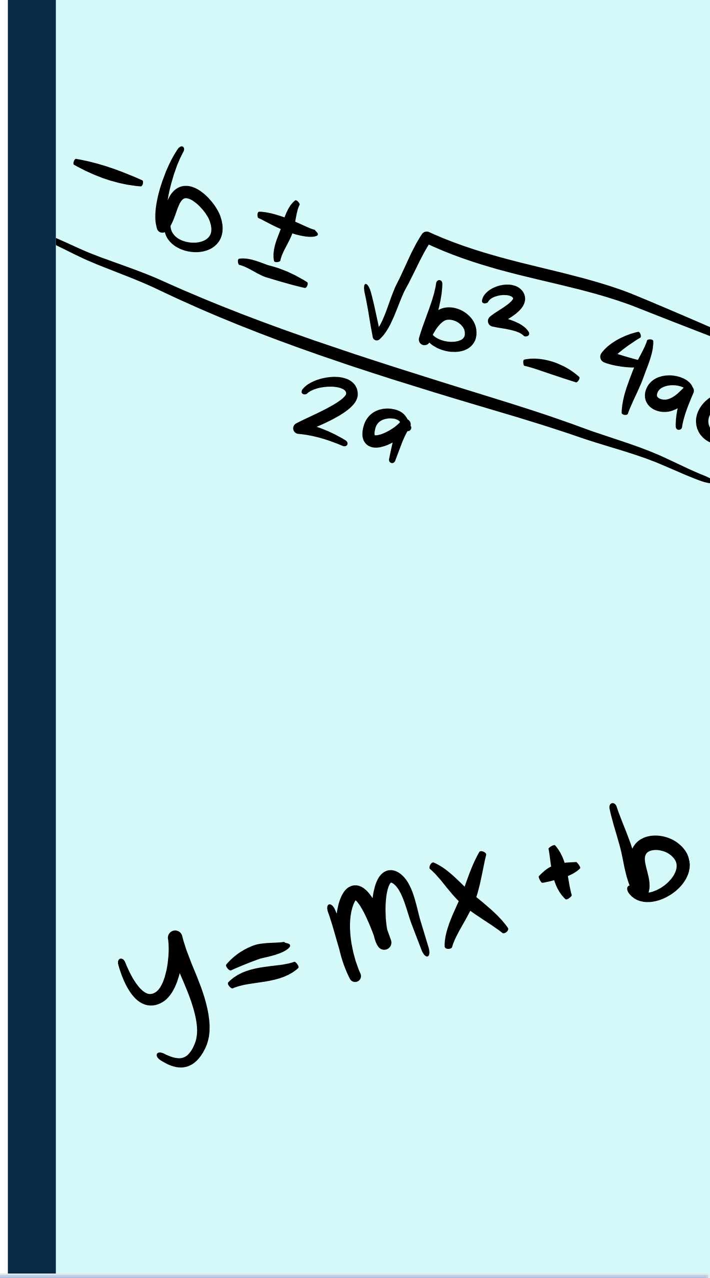
$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$y = mx + b$$



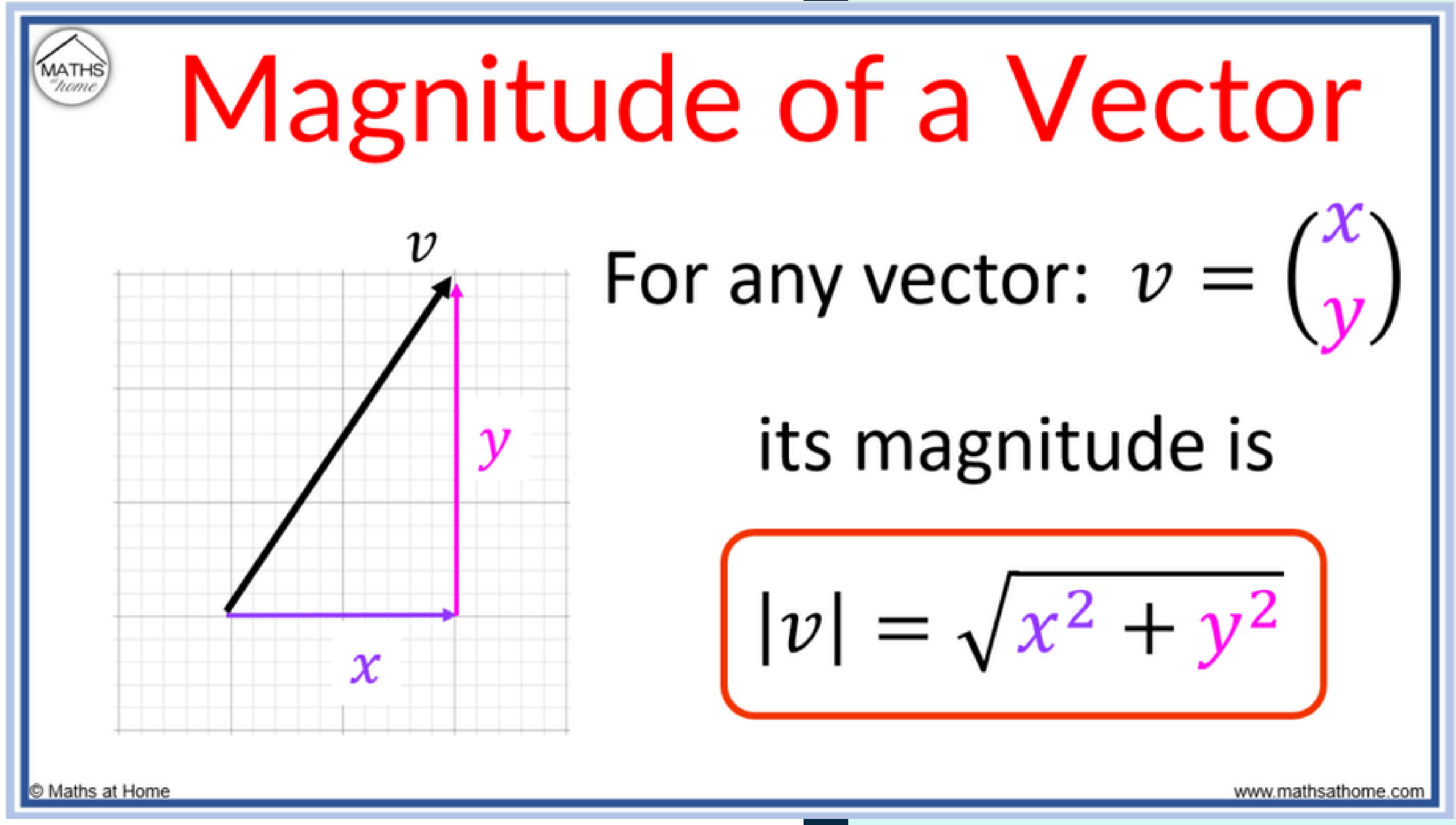

$$V = \frac{4}{3}\pi r^3$$

# 9.4 - Defining and Differentiating Vector-Valued Functions




Vector basics:

- Vectors have magnitude (length) and direction.
- Vectors can be represented by directed line segments.
- Vectors are equal if they have the same direction and magnitude.
- Magnitude is designated by $||v||$
- Vectors have a horizontal and vertical component.
- Component form of a vector is $\langle x, y \rangle$



# 9.4 - Defining and Differentiating Vector-Valued Functions

Finding the derivative of a vector-valued function is pretty straightforward. Suppose a vector-valued function is defined as $u(t) = (v(t), w(t))$, then its derivative is the vector-valued function $u'(t) = (v'(t), w'(t))$.

**Properties of the derivative for vector-valued functions**

$$\frac{d}{dt}[c \cdot r(t)] = c \cdot r'(t)$$

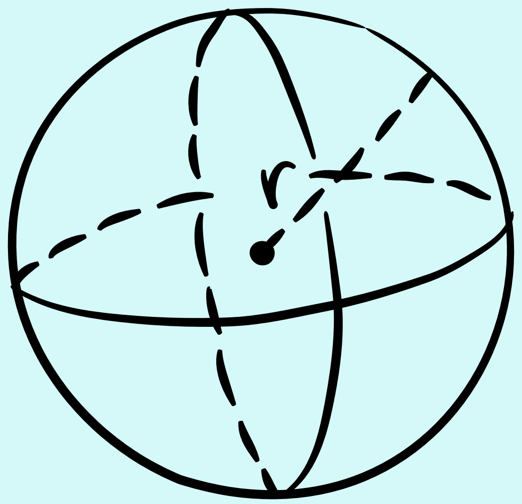
$$\frac{d}{dt}[r(t) \pm s(t)] = r'(t) \pm s'(t)$$

$$\frac{d}{dt}[r(t) \cdot s(t)] = r'(t)s(t) + r(t)s'(t)$$

$$\frac{d}{dt}[r(s(t))] = r'(s(t)) \cdot s'(t)$$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.4 - Defining and Differentiating Vector-Valued Functions

Let $g$ be a vector-valued function defined by
$g(t) = (-2\sin(t+1), 5t^2 - 2t)$.

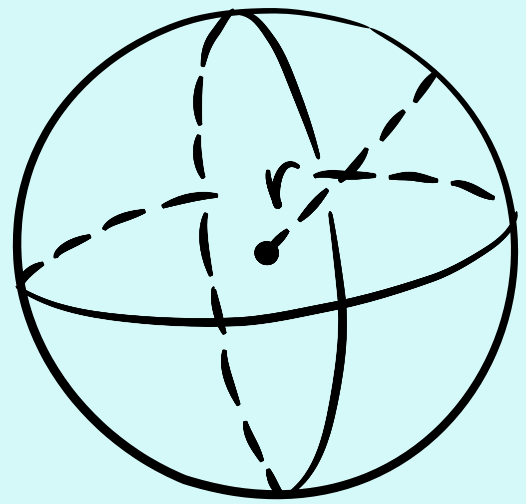
Find $g'(t)$.

$$g'(t) = \left( \frac{d}{dt}(-2\sin(t+1)), \frac{d}{dt}(5t^2-2t) \right)$$

$$= (-2\cos(t+1), 10t-2)$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

# 9.4 - Defining and Differentiating Vector-Valued Functions

Let $h$ be a vector-valued function defined by

$$h(t) = \left(-\frac{3}{t+2}, e^{3t}\right).$$

Find $h$'s *second* derivative $h''(t)$.

$$h'(t) = \left(\frac{d}{dt}\left(-\frac{3}{t+2}\right), \frac{d}{dt}\left(e^{3t}\right)\right)$$

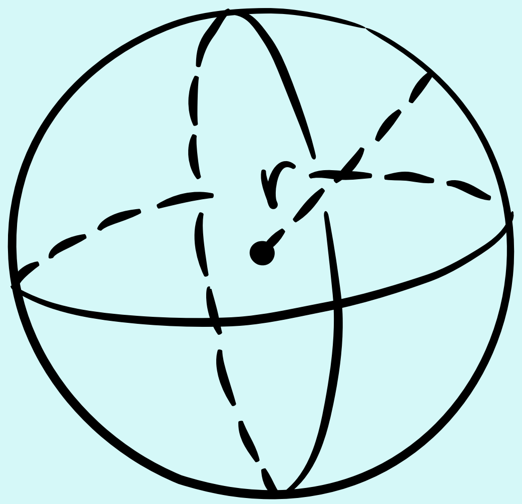
$$= \left(\frac{3}{(t+2)^2}, 3e^{3t}\right)$$

$$h''(t) = \left(\frac{d}{dt}\left(\frac{3}{(t+2)^2}\right), \frac{d}{dt}\left(3e^{3t}\right)\right)$$

$$= \boxed{\left(-\frac{6}{(t+2)^3}, 9e^{3t}\right)}$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

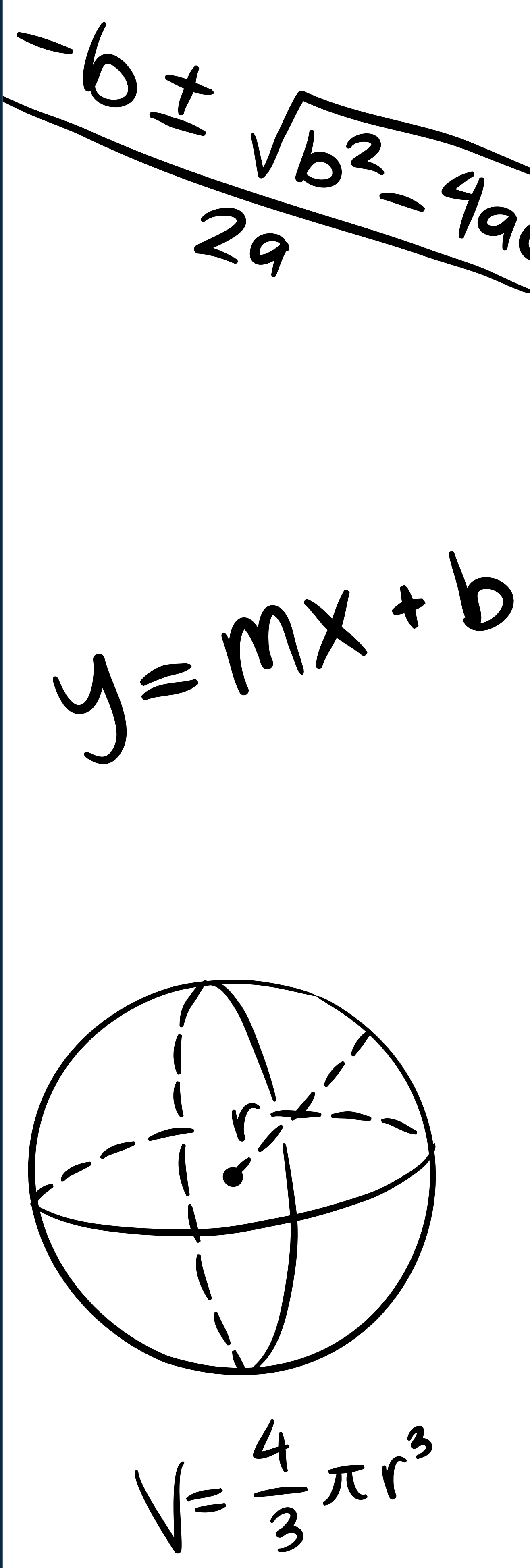
# 9.5 - Integrating Vector Valued Functions

**Integration of Vector-Valued Functions**

If $r(t) = \langle f(t), g(t) \rangle$ then

$$\int r(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt \rangle$$

**Don't forget +C!**



# 9.5 - Integrating Vector Valued Functions

Find the vector-valued function f(t) that satisfies the initial given conditions:

$f'(0) = \langle 3, 0\rangle, f(0) = \langle 0, 3\rangle,$
$f''(t) = \langle 5\cos t, -2\sin t\rangle$

$f'(t) = \langle \int 5\cos t\, dt, \int -2\sin t\, dt\rangle$ ① integrate

$= \langle 5\sin t + C_1, 2\cos t + C_2\rangle$ ② use initial condition to find $C_{1,2}$

since $f'(0) = \langle 3, 0\rangle$, $5\sin(0) + C_1 = 3$ $\quad 2\cos(0) + C_2 = 0$

③ plug back in $C_1 = 3$ $\quad C_2 = -2$

$f'(t) = \langle 5\sin t + 3, 2\cos t - 2\rangle$ ④ repeat to get f(t)

$f(t) = \langle \int (5\sin t + 3)\, dt, \int (2\cos t - 2)\, dt\rangle$

$f(t) = \langle -5\cos t + 3t + C_3, 2\sin t - 2t + C_4\rangle$

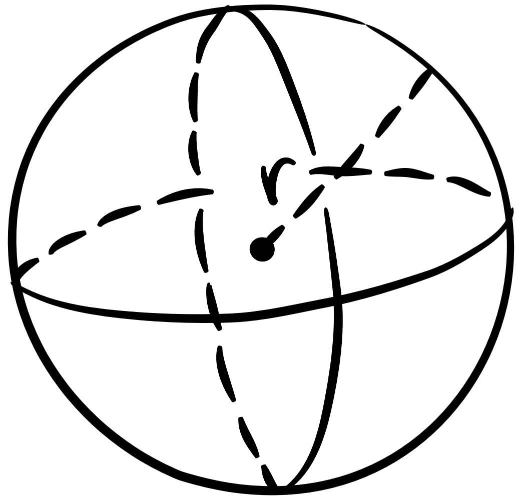
since $f(0) = \langle 0, 3\rangle$, $-5\cos(0) + 3(0) + C_3 = 0$ $\quad 2\sin(0) - 2(0) + C_4 = 3$

$C_3 = 5$ $\quad C_4 = 3$

$f(t) = \langle -5\cos t + 3t + 5, 2\sin t - 2t + 3\rangle$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.6 - Solving Motion Problems Using Parametric and Vector-Valued Functions

<u>Position</u>: $r(t) = \langle x(t), y(t) \rangle$

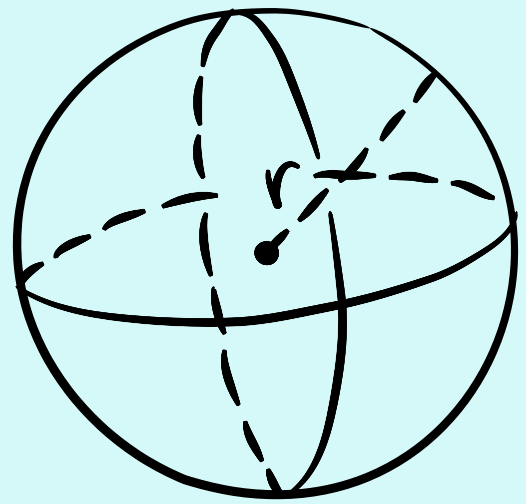
<u>Velocity</u>: $v(t) = r'(t) = \langle x'(t), y'(t) \rangle$

<u>Acceleration</u>: $a(t) = r''(t) = \langle x''(t), y''(t) \rangle$

<u>Speed</u>: $\|v(t)\| = \|r'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$



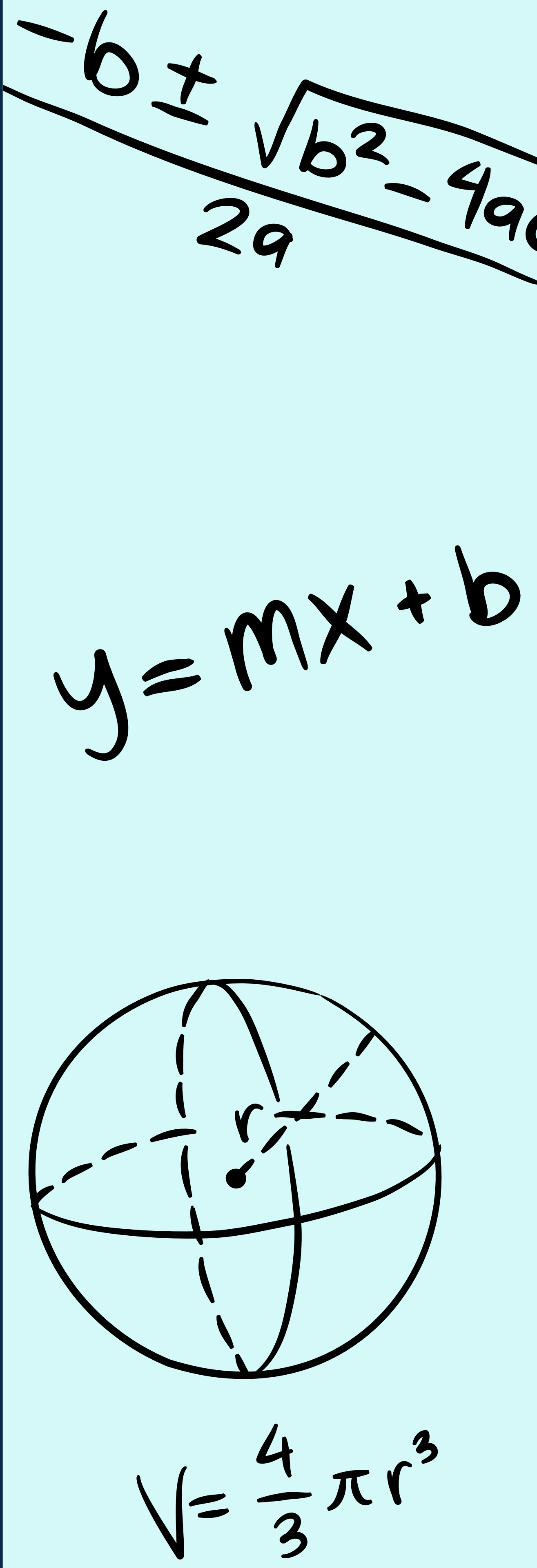

$V = \frac{4}{3}\pi r^3$

# 9.6 - Solving Motion Problems Using Parametric and Vector-Valued Functions

Distance Traveled: $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

**Speeding up:** Acceleration and velocity have the same sign

**Slowing down:** Acceleration and velocity have different signs



# 9.6 - Solving Motion Problems Using Parametric and Vector-Valued Functions

$x(t) = 2\sin\frac{t}{2}$ and $y(t) = 2\cos\frac{t}{2}$ for time $t > 0$.

Find the speed of the particle.

Speed: $\|v(t)\| = \|r'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$

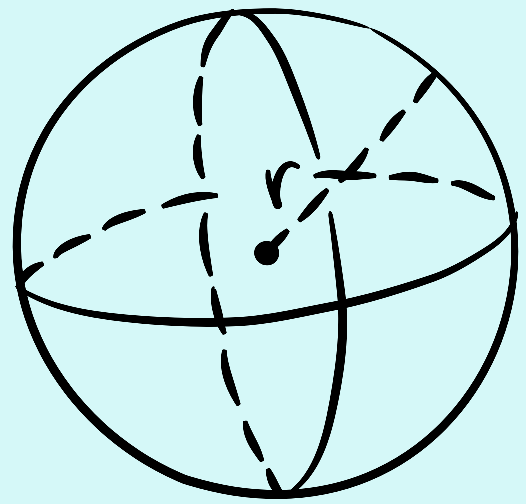
$x'(t) = \cos\left(\frac{t}{2}\right)$ $\quad y'(t) = -\sin\left(\frac{t}{2}\right)$

$\text{speed} = \sqrt{\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right)}$ ← pythagorean identity

$= \sqrt{1} = \boxed{1}$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.6 - Solving Motion Problems Using Parametric and Vector-Valued Functions

At time $t \geq 0$, a particle moving in the $xy$-plane has velocity vector given by $v(t) = \langle t^3, 4t \rangle$. What is the acceleration vector when $t = 2$?

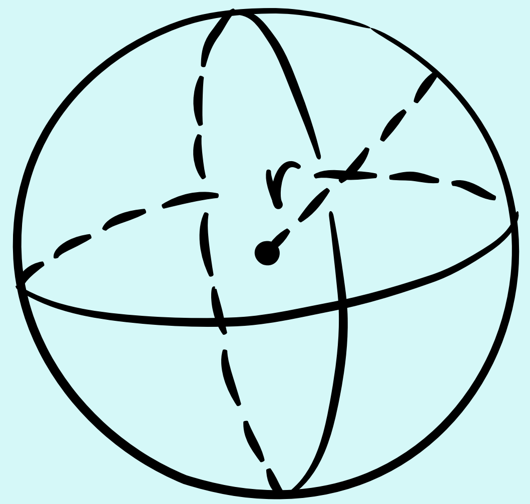
$$a(t) = \frac{d}{dt}\left(v(t)\right) = \left\langle \frac{d}{dt}(t^3), \frac{d}{dt}(4t) \right\rangle$$

$$= \langle 3t^2, 4 \rangle$$

$$a(2) = \langle 3(4), 4 \rangle = \langle 12, 4 \rangle$$

$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

# 9.7 - Defining Polar Coordinates and Differentiating in Polar Form

$(x, y)$ is for a **rectangular** coordinate system.

$(r, \theta)$ is for a **polar** coordinate system.

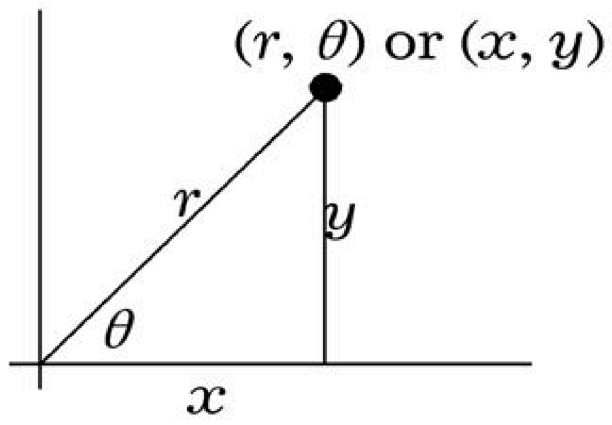
$r$ is a directed distance from the origin to a point P.

$\theta$ is the directed angle

## CONVERTING BETWEEN RECTANGULAR AND POLAR COORDINATES

- Polar coordinates to rectangular coordinates

  $x = r\cos\theta; \quad y = r\sin\theta$




- Rectangular coordinates to polar coordinates

  $r^2 = x^2 + y^2; \quad \tan\theta = \dfrac{y}{x}$

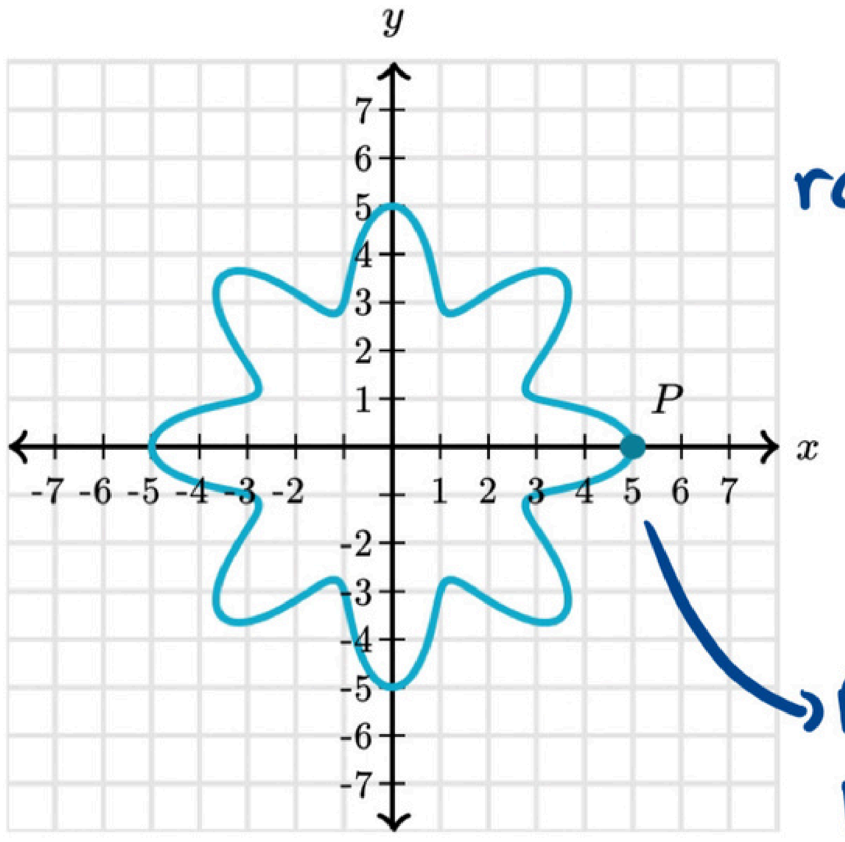
$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$

$+b$

$r^3$

# 9.7 - Defining Polar Coordinates and Differentiating in Polar Form

Let $r$ be the polar function $r(\theta) = \cos(8\theta) + 4$. Here is its graph for $0 \le \theta \le 2\pi$:




since $x = r\cos\theta$,

$r = \cos(8\theta) + 4$ multiply both sides by $\cos\theta$

$r\cos\theta = \cos(8\theta)\cos\theta + 4\cos\theta$

$x = \cos(8\theta)\cos\theta + 4\cos\theta$

$\frac{dx}{d\theta} = -8\sin(8\theta)\cos\theta - \cos(8\theta)\sin(\theta) - 4\sin\theta$

$\left.\frac{dx}{d\theta}\right|_{\theta=0} = -8\sin(0)\cos(0) - \cos(0)\sin(0) - 4\sin(0)$

$\boxed{= 0}$

Point P occurs at $\theta = 0$ or $\theta = 2\pi$ (since $0 \le \theta \le 2\pi$)

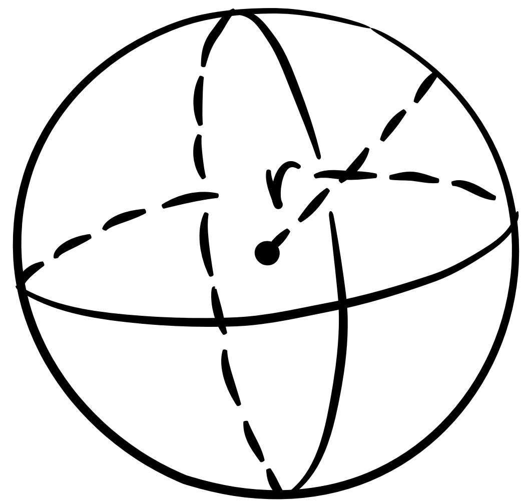
$r(0) = \cos(0) + 4 = 5$ Point $P = (5, 0)$ or $(5, 2\pi)$

$r(2\pi) = \cos(2\pi) + 4 = 5$

What is the rate of change of the $x$-coordinate with respect to $\theta$ at the point $P$?

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# 9.7 - Defining Polar Coordinates and Differentiating in Polar Form

Consider the polar curve $r = \cos(\theta)$.

**What is the slope of the tangent line to the curve $r$ when $\theta = \frac{\pi}{4}$?**

*Give an exact expression.*

Since $x = r\cos\theta$, $y = r\sin\theta$,

$r\cos\theta = \cos^2\theta$

$r\sin\theta = \cos\theta\sin\theta$

$x = \cos^2\theta$

$y = \cos\theta\sin\theta$

$$\frac{dx}{d\theta} = -2\cos\theta\sin\theta$$

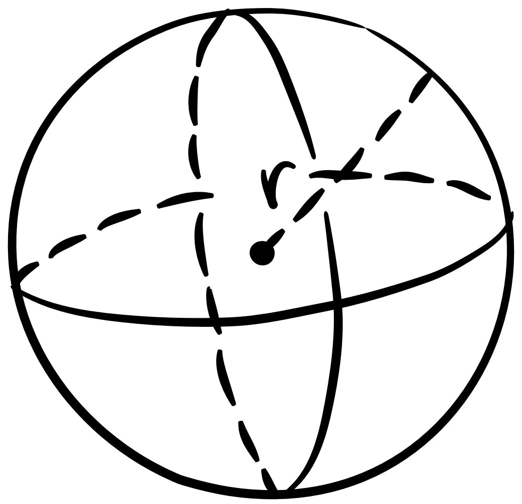
$$\frac{dy}{d\theta} = \cos^2\theta - \sin^2\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos^2\theta - \sin^2\theta}{-2\cos\theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2}{-2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} = \boxed{0}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$




$$V = \frac{4}{3}\pi r^3$$

# 9.8 - Area Bounded by a Single Polar Curve

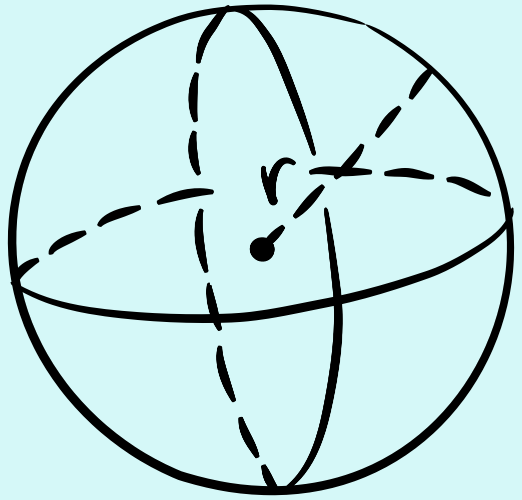
This is the formula for the area enclosed by a polar curve $r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$:

$$\int_{\alpha}^{\beta} \frac{1}{2}(r(\theta))^2 d\theta$$

Be careful finding the interval of integration!

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = mx + b$$



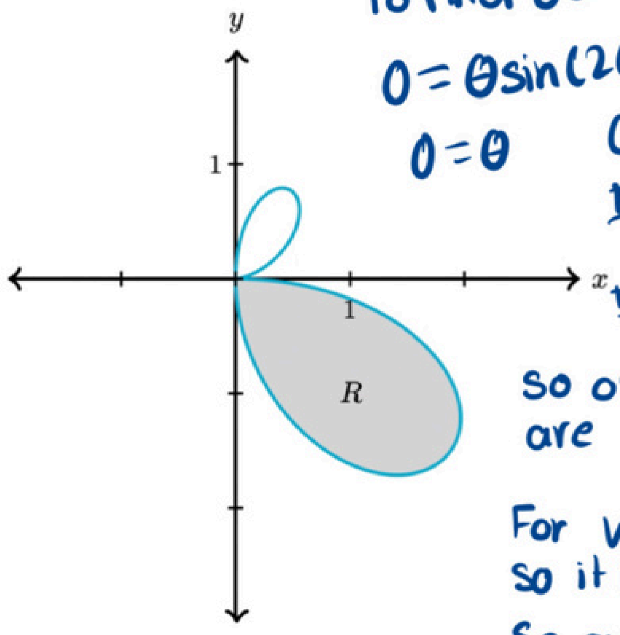

$$V = \frac{4}{3}\pi r^3$$

# 9.8 - Area Bounded by a Single Polar Curve

The polar curve $r(\theta) = \theta \cdot \sin(2\theta)$ is graphed for $0 \le \theta \le \pi$.

Let $R$ be the region in the fourth quadrant enclosed by the curve and the $x$-axis.

To find our bounds, we set $r = 0$




$0 = \theta\sin(2\theta)$

$0 = \theta \qquad 0 = \sin(2\theta)$

$\frac{\pi n}{2} = \frac{2\theta}{2}$ — solving trig equation for an integer $n$

$\frac{\pi n}{2} = \theta$

So our $\theta$ values between $0 \le \theta \le \pi$ are $\theta = 0, \frac{\pi}{2}, \pi$

For values between $\frac{\pi}{2}$ and $\pi$, $r$ is negative so it is in Quadrant IV even if the angle is in Quadrant II

So, our $\alpha = \frac{\pi}{2}$ and $\beta = \pi$

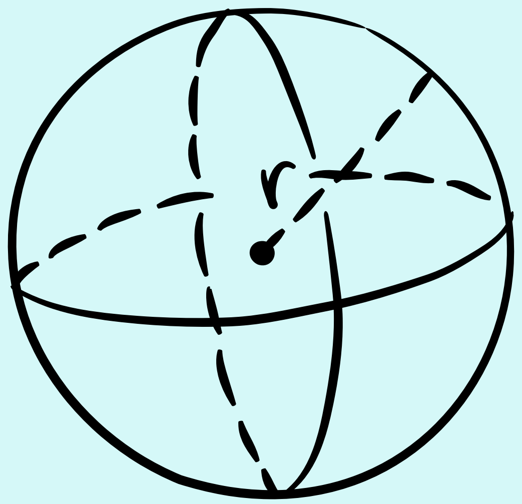
**Which integral represents the area of $R$?**

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta \Rightarrow \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\theta\sin(2\theta))^2\,d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\theta^2\sin^2(2\theta))\,d\theta$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$

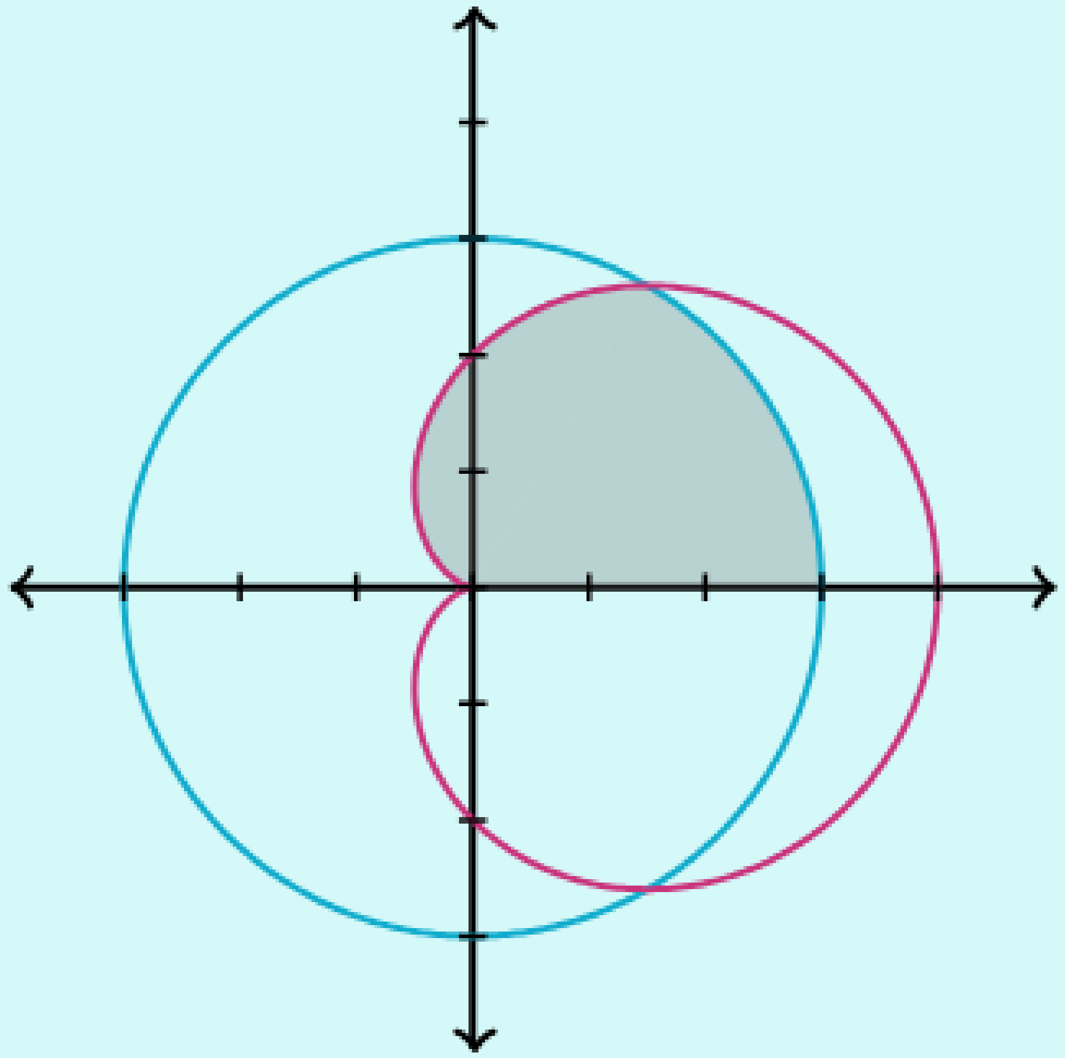


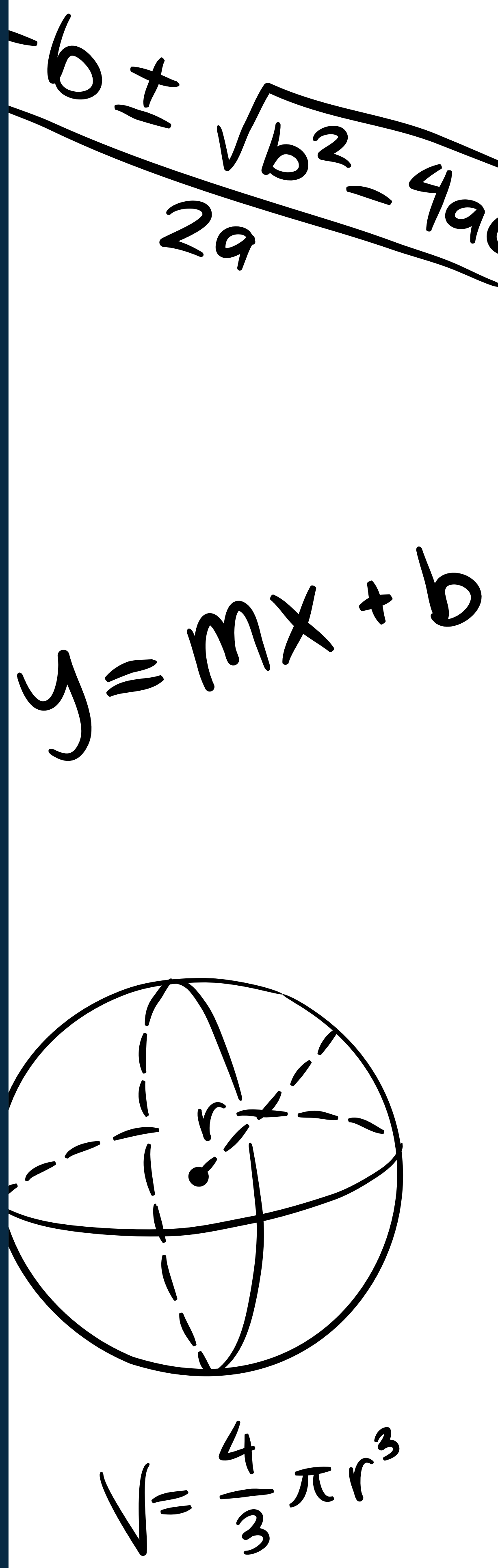

$$V = \frac{4}{3}\pi r^3$$

# 9.9 Area Bounded by Two Polar Curves

Continue using the area formula to subtract big area - small area

Watch out for points of intersection (you can get them by setting both polar equations equal to each other) and symmetry



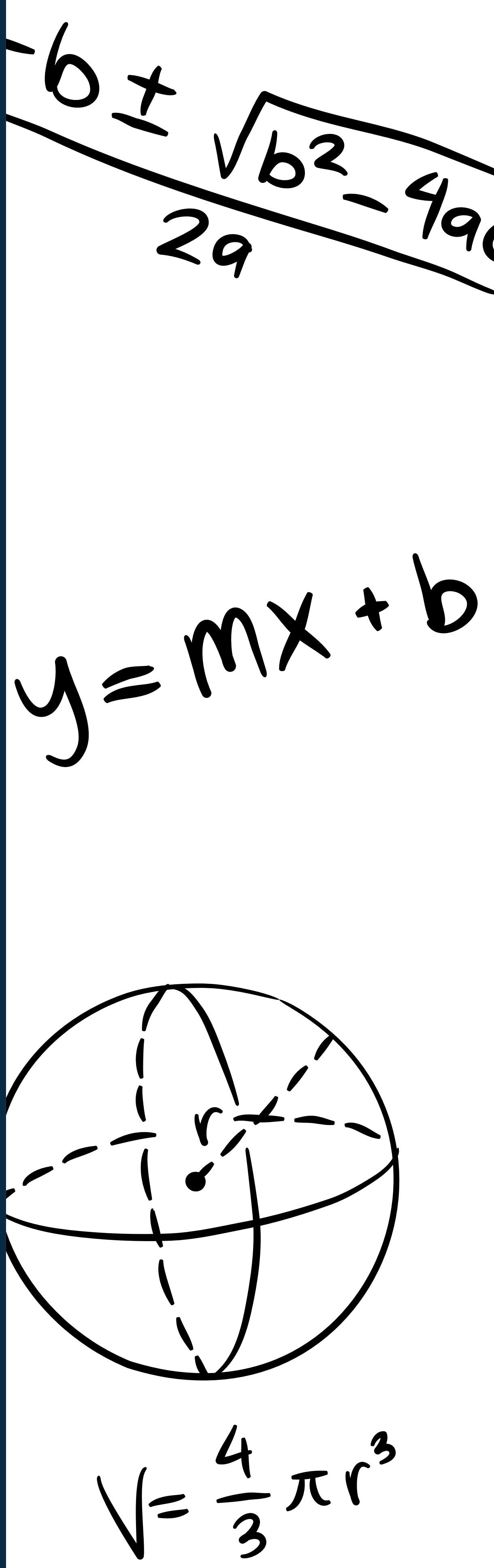


# 9.9 Area Bounded by Two Polar Curves

## Area Bounded by Two Polar Curves

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_2)^2 \, d\theta - \frac{1}{2}\int_{\alpha}^{\beta} (r_1)^2$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} ({r_2}^2 - r_1^2) \, d\theta$$



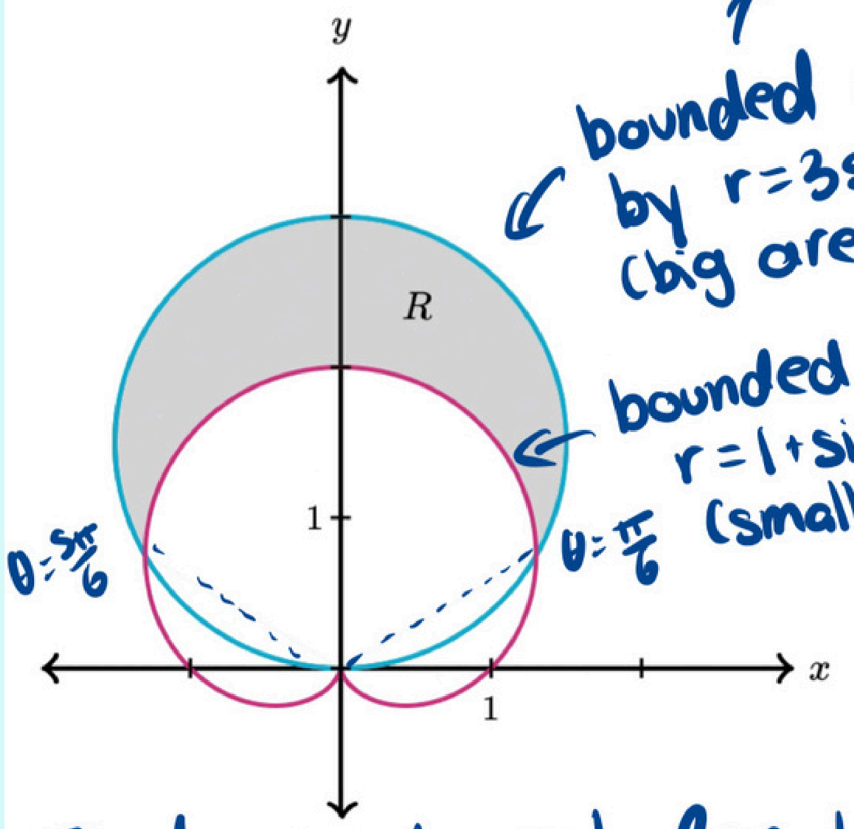
# 9.9 Area Bounded by Two Polar Curves

Let $R$ be the region that is inside the polar curve $r = 3\sin(\theta)$ and outside the polar curve $r = 1 + \sin(\theta)$, as shown in the graph. The curves intersect at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} (r_1^2 - r_2^2)\, d\theta$$

big area - small area

$$\boxed{\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((3\sin\theta)^2 - (1+\sin(\theta))^2\right) d\theta}$$



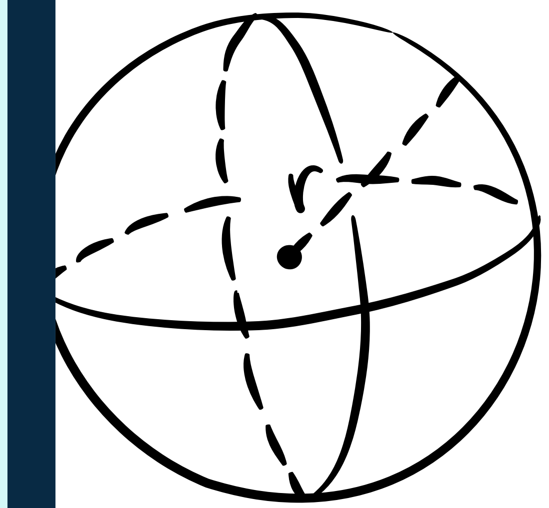

bounded above by $r = 3\sin(\theta)$ (big area)

bounded below by $r = 1 + \sin(\theta)$ (small area)

Find an integral for the area

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$y = mx + b$




$V = \frac{4}{3}\pi r^3$

# Misc

**Arc Length of Polar Curve:**

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

# GOOD LUCK!

**Don’t give up, you’re almost there :)**

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