DIFFERENTIAL EQUATIONS

AP CALC UNIT 7

loopsofkindness.com
Made with love by Diana Moya :)

TABLE OF CONTENTS

7.1 MODELING SITUATIONS WITH DIFFERENTIAL EQUATIONS

<u>Directly Proportional</u> (usually we just say "proportional")

If a is proportional to b, then $a = kb$, where k is a constant.

<u>Inversely Proportional</u> (sometimes we say "proportional to the reciprocal")

If a is INVERSELY proportional to b, then $a = \frac{k}{b}$, where k is a constant.

Differential equations are those that have a differential (derivative) in an equation!

Differential (derivative)

equation

$$\frac{dy}{dx} + x = 6$$

7.1 MODELING SITUATIONS WITH DIFFERENTIAL EQUATIONS

The rate of change of the perceived stimulus p with respect to the measured intensity s of the stimulus is inversely proportional to the intensity of the stimulus.

reciprocal

Which equation describes this relationship?

Rate of change of p with respect to s $= \frac{dp}{ds}$

$$\frac{dp}{ds} = \frac{k}{s}$$

k ← constant of proportionality

s ← intensity

7.1 MODELING SITUATIONS WITH DIFFERENTIAL EQUATIONS

Each month the balance, B, of Harper's loan increases by 0.22% and decreases by $\$250.00$.

Which equation describes this relationship?

$$\frac{dB}{dt} = 0.0022B - 250$$

increases (changes) by 0.22% of B

decreases by 250

Remember, we are looking for change in the balance, not the actual balance

Increase & decrease every month: we are looking for change in balance B with respect to months t

$\frac{dB}{dt}$

you can use any variable except B for months techically, but we usually represent time w/ t and the answer choices (not pictured) are in terms of t

7.2 VERIFYING SOLUTIONS FOR DIFFERENTIAL EQUATIONS

We can verify solutions to differential equations by finding the derivative of the solution and plugging in to the original differential equation.

Remember: Solutions to differential equations are just functions $y = f(x)$ that satisfy the differential equation when f and its derivatives are substituted back into the differential equation.

7.2 VERIFYING SOLUTIONS FOR DIFFERENTIAL EQUATIONS

$$\frac{dy}{dx} = -\frac{x}{y}$$

Is $y = \sqrt{10 - x}$ a solution to the above equation?

$y = (10-x)^{\frac{1}{2}}$ $\quad \frac{dy}{dx} = \frac{1}{2\sqrt{10-x}}$

① Find the derivative of y

$-\frac{1}{2\sqrt{10-x}} = \frac{-x}{y}$

② Plug into the differential equation, as if it were a solution

$-\frac{1}{2\sqrt{10-x}} = -\frac{x}{\sqrt{10-x}}$

③ Plug in the y

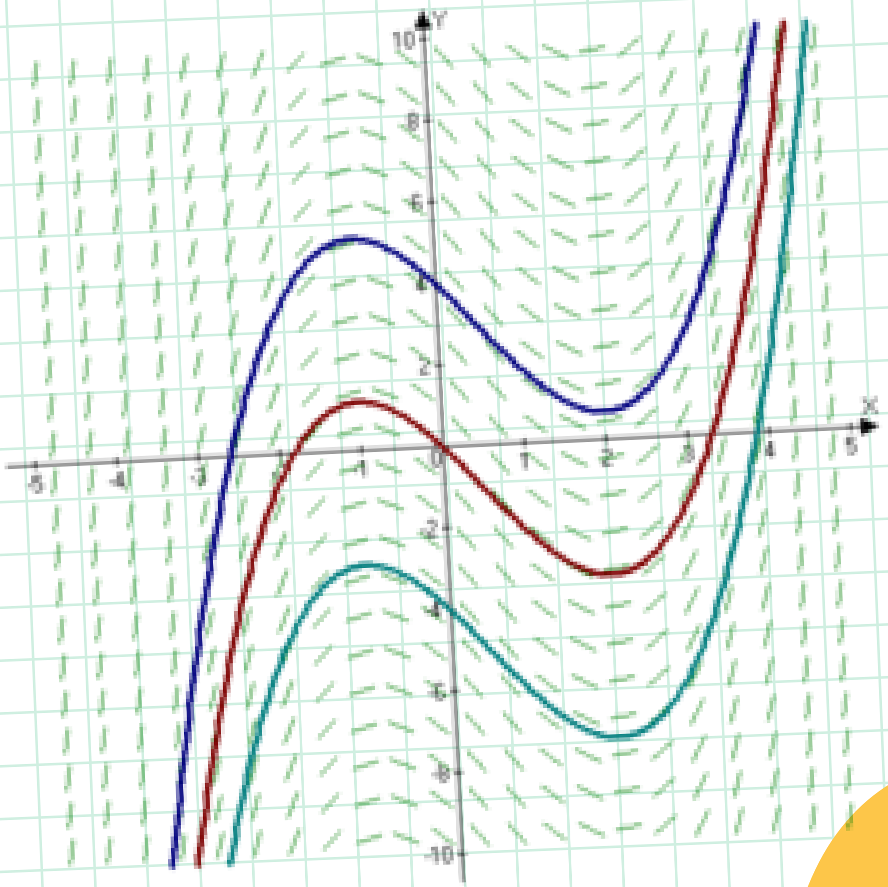
Since $-\frac{1}{2\sqrt{10-x}} \neq -\frac{x}{\sqrt{10-x}}$, $y = \sqrt{10-x}$ is not a solution.

7.3 SKETCHING SLOPE FIELDS

Slope fields are a graphical representation of the solutions to a differential equation

They are a great tool for visualizing differentials!

To sketch a slope field, you must sketch a short line segment representing the slope (dy/dx) at each point (x, y).



7.3 SKETCHING SLOPE FIELDS

In drawing the slope field for the differential equation $\frac{dy}{dx} = x + 2y - 2$, I would place short line segments at select points on the xy-plane.

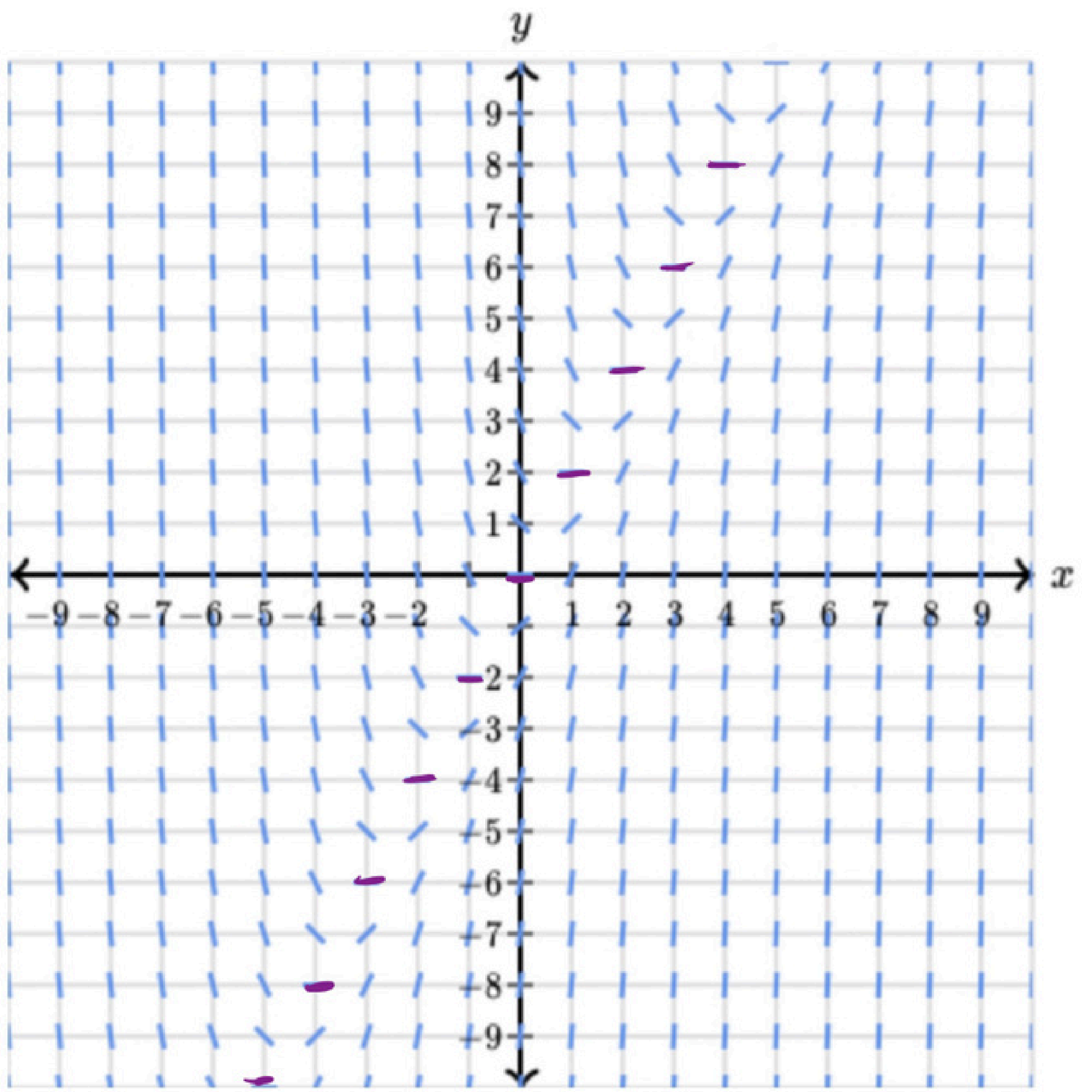
Complete the sentences.

At the point $(-2, 0)$, I would draw a short segment of slope -4. $\frac{dy}{dx} = -2 + 2(0) - 2 = -4$

At the point $(0, 3)$, I would draw a short segment of slope 4. $\frac{dy}{dx} = 0 + 6 - 2 = 4$

At the point $(1, 1)$, I would draw a short segment of slope 1. $\frac{dy}{dx} = 1 + 2 - 2 = 1$

Which differential equation generates the slope field?



① Recognize that $\frac{dy}{dx} = 0$ along all segments that appear along the line $y = 2x$. Making the substitution that $\frac{dy}{dx} = 0$ when $y = 2x$ into the following answer choices

Ⓐ $\frac{dy}{dx} = x + 2y$

Ⓑ $\frac{dy}{dx} = 2x + y$

CORRECT (SELECTED)
$\frac{dy}{dx} = 2x - y$ $0 \overset{\checkmark}{=} 2x - 2x$

Ⓓ $\frac{dy}{dx} = \frac{2x}{y}$

Ⓔ $\frac{dy}{dx} = -\frac{2x}{y}$

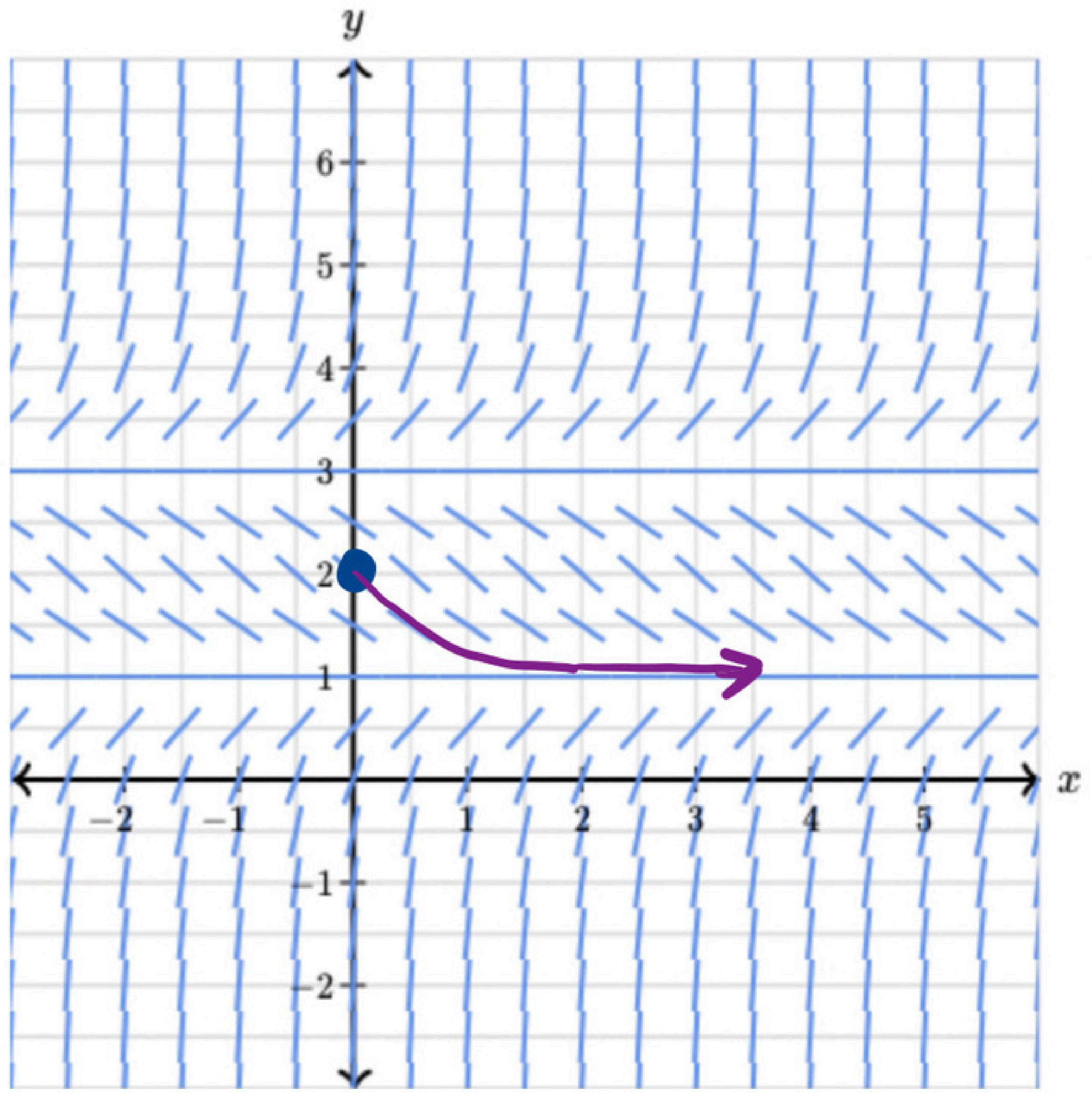
7.4 REASONING USING SLOPE FIELDS

You can match slope fields of differential equations to solutions by tracing out a curve using the slope field as a guide.

If the question asks you to find the solution curve passing through a specific point, start at that point and follow the shape of the slope field on both sides to sketch the solution curve.

7.4 REASONING USING SLOPE FIELDS

If the initial condition is $(0, 2)$, what is the range of the solution curve $y = f(x)$ for $x \geq 0$?



① Plot the initial condition

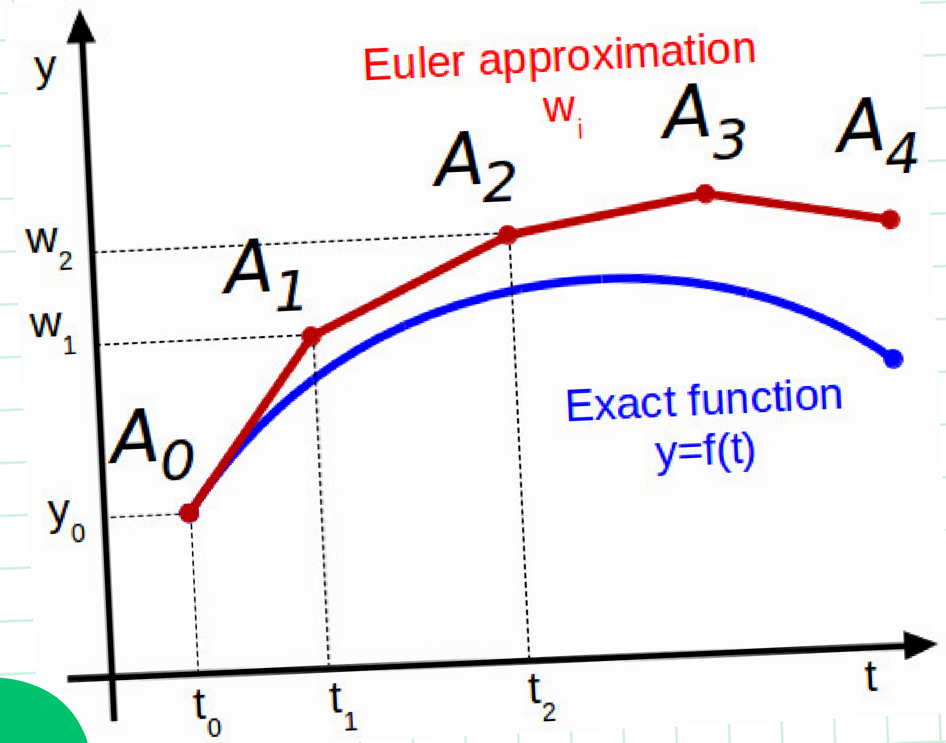
② Trace a line following the slope field as a guide

(be careful to follow asymptotes and initial conditions!)

7.5 APPROXIMATING SOLUTIONS USING EULER'S METHOD

BC ONLY!!!

Euler's method is rough to explain with a couple slides (trust me I tried), so here are some links that explain the concept well:



[Flipped Math]

[Khan]

[Organic Chemistry Tutor]

7.6 GENERAL SOLUTIONS USING SEPARATION OF VARIABLES

Separating the variables to different sides in differential equations allows us to integrate to find a general solution!

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\Rightarrow \frac{dy}{g(y)} = f(x)\,dx$$

$$\Rightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$\int h(y)\,dy = \int g(x)\,dx$$

Separable differential equations are written in the form:

$$\frac{dy}{dx} = f(x)g(y) \text{ or } \frac{dy}{dx} = \frac{f(x)}{g(x)}$$

7.6 GENERAL SOLUTIONS USING SEPARATION OF VARIABLES

Solve the equation.

$$\frac{dy}{dx} = 8x^3y - 8xy$$

① Separate the x's and y's by factoring

$$\frac{dy}{dx} = y(8x^3 - 8x)$$

② Rearrange to move each variable to one side

$$\frac{dy}{y} = (8x^3 - 8x)dx$$

③ Integrate both sides

$$\int \frac{dy}{y} = \int (8x^3 - 8x)dx$$

$$\ln y = 2x^4 - 4x^2 + C_1$$ ← don't forget +C!

$$y = e^{2x^4 - 4x^2 + C_1} = C_2 e^{2x^4 - 4x^2}$$ ← solve using properties of logarithms to bring y down and properties of exponents to bring the arbitrary constant down

Remember: $e^{\ln y} = y$

$x^{a+b} = x^a x^b$

7.7 PARTICULAR SOLUTIONS USING INITIAL CONDITIONS AND SEPARATION OF VARIABLES

Instead of leaving +C as an arbitrary constant, we figure out what +C is using initial conditions!

Since our general solution after separation of variables is already in the form $y = f(x) + C$, we just have to plug in an initial condition (x, y) and solve for C to get the solution for that initial condition.

7.7 PARTICULAR SOLUTIONS USING INITIAL CONDITIONS AND SEPARATION OF VARIABLES

$\frac{dy}{dt} = 3t^2 + 1$ and $y(1) = 5$.

What is t when $y = 3$?

① Separable differential equation

$dy = (3t^2+1)\,dt \Rightarrow \int dy = \int (3t^2+1)\,dt$

$y = t^3 + t + C$

② Use initial condition

$5 = 1 + 1 + C \qquad C = 3$

③ Substitute for C to get particular solution

$y = t^3 + t + 3$

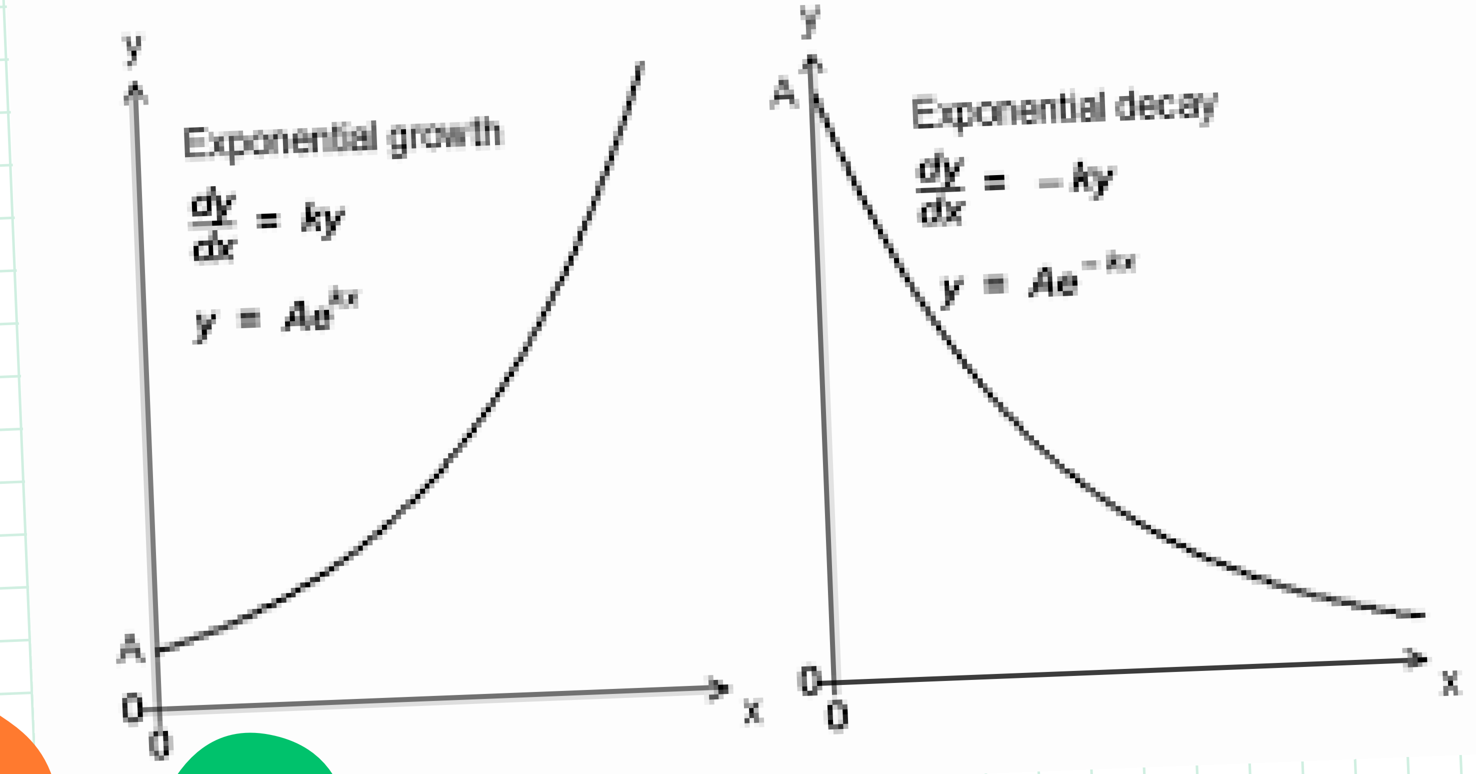
④ Solve for t when $y = 3$

$3 = t^3 + t + 3$

$0 = t^3 + t$, $\boxed{t = 0}$

7.8 EXPONENTIAL MODELS WITH DIFFERENTIAL EQUATIONS

We can use differential equations to model the rate of change of an exponential function!



The general solution of equations of the form $\frac{dy}{dt} = ky$ is

$$y = C \cdot e^{kt}$$

for some constant C.

This can be found using separation of variables. Got it, thanks!

$$\frac{df}{dx} = kf$$

$$\frac{df}{f} = kdx$$

$$\int \frac{df}{f} = \int kdx$$

$$\ln(|f|) = kx + c$$

$$e^{\ln(|f|)} = e^{kx+c}$$

$$f = C \cdot e^{kx} \quad \text{Let } C = e^{c} \geq 0$$

7.8 EXPONENTIAL MODELS WITH DIFFERENTIAL EQUATIONS

During one time period, the price of rhodium increased at a rate that was proportional to the price of rhodium at that time.

direct proportionality

The price for an ounce of rhodium was $475 initially, and it quadrupled every 25 months.

What was the price for an ounce of rhodium after 18 months?

① Price of rhodium = P
time in months = t

$$\frac{dP}{dt} = kP$$

(solution (try solving using separation of variables to see why!))

$$P = Ce^{kt}$$

Let's find what C and k are.

② Since $P(0)=475$, $475 = Ce^0 \Rightarrow 475 = C$

We now have $P = 475e^{kt}$. Only missing k now!

③ Since the price quadruples every 25 months, we have:

Price in 25 mo

$$\frac{P(t+25)}{P(t)} = 4 \Rightarrow \frac{475e^{k(t+25)}}{475e^{kt}} = 4$$

$$\frac{e^{kt+25k}}{e^{kt}} = 4 \Rightarrow e^{kt+25k-kt} = 4$$

$$e^{25k} = 4$$

(take ln of both sides to get rid of e)

$$\frac{25k}{25} = \frac{\ln(4)}{25}$$

$$k = \frac{\ln(4)}{25}$$

④ So, the equation can be represented as

$$y = 475e^{\frac{\ln(4)}{25}t}$$

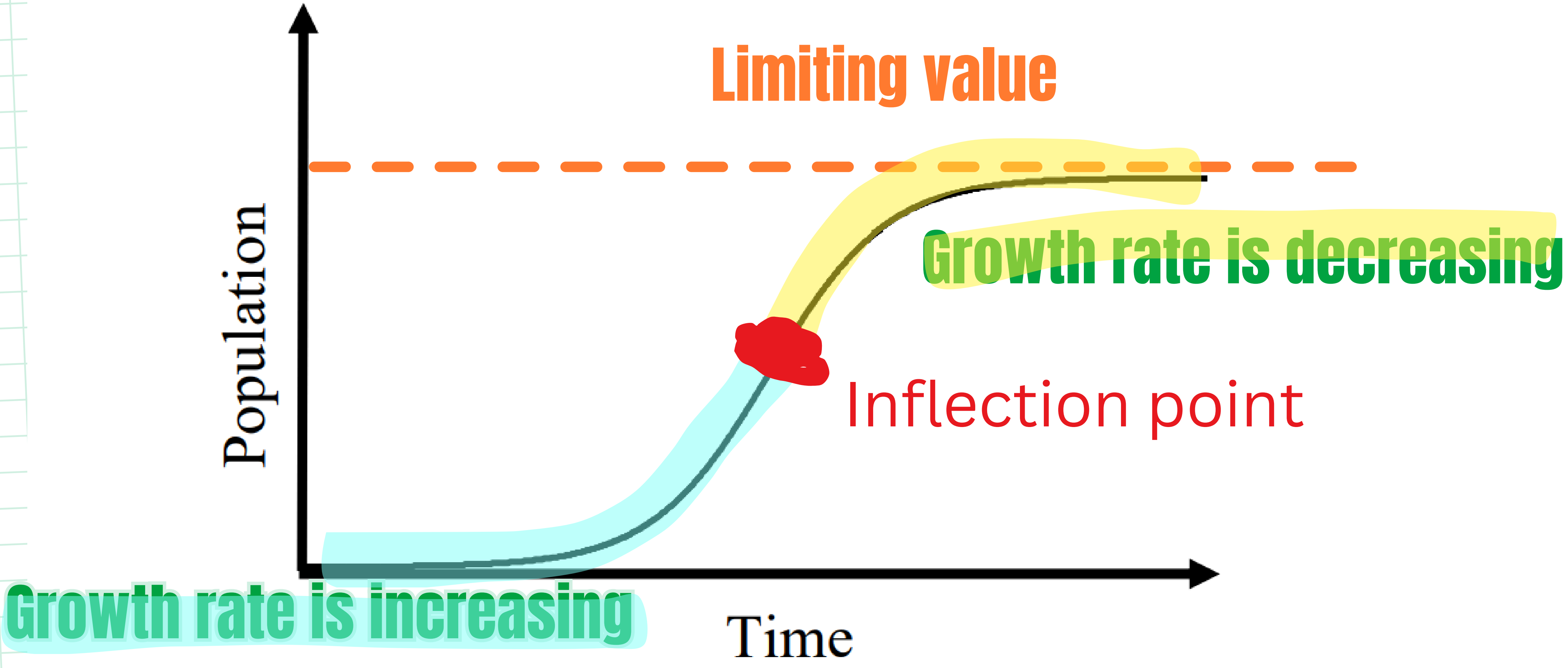
When $t=18$, $y = 475e^{\frac{\ln(4)}{25}(18)}$

$$\approx 1289$$

BC ONLY - 7.9 LOGISTIC MODELS WITH DIFFERENTIAL EQUATIONS

Logistic population growth is when the growth rate increases quickly at first, but then slows as the population reaches carrying capacity.

Graphs would look something like this:



BC ONLY - 7.9 LOGISTIC MODELS WITH DIFFERENTIAL EQUATIONS

Logistic Differential Equation

The derivative of a logistic function is typically written in one of the following forms:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

or if you manipulate this algebraically you could see it as

$$\frac{dy}{dt} = \frac{k}{L}y(L - y)$$

In either form, k and L are positive constants and L is the limiting value.

BC ONLY - 7.9 LOGISTIC MODELS WITH DIFFERENTIAL EQUATIONS

The number $P(t)$ of people who have heard about a certain contest after t weeks satisfies the following logistic differential equation:

$$\frac{dP}{dt} = \frac{1}{50} P \cdot (35{,}000 - P)$$

Initially, 1000 people have heard about the contest.

What is the carrying capacity of the population of people who have heard about the contest?

① Get it in the form

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{L}\right)$$

where L is the limiting factor (carrying capacity)

$$\frac{dP}{dt} = \frac{1}{50} \cdot \frac{1}{35000} P\left(1 - \frac{P}{35000}\right)$$

factor out 35000

35000 is the L (carrying capacity)


THANK YOU!
Please leave a comment or send someone else the resource if you found this helpful!