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Integration and Accumulation of Change

AP CALC - UNIT 6

Content

6.1 Exploring Accumulation of Change

- The area of the region between the graph of a rate of change function and the x-axis gives the accumulation of change
- In some cases, accumulation of change (area between curve and x-axis) can be evaluated using geometry to split it up into simpler shapes and adding up the area of those shapes
- If the curve is above the x-axis, the accumulation of change is positive. If the curve is below the x-axis, the accumulation of change is negative.
- You can find the unit for accumulation of change (area between curve and x-axis) by multiplying (unit for rate of change)(unit on the x-axis)
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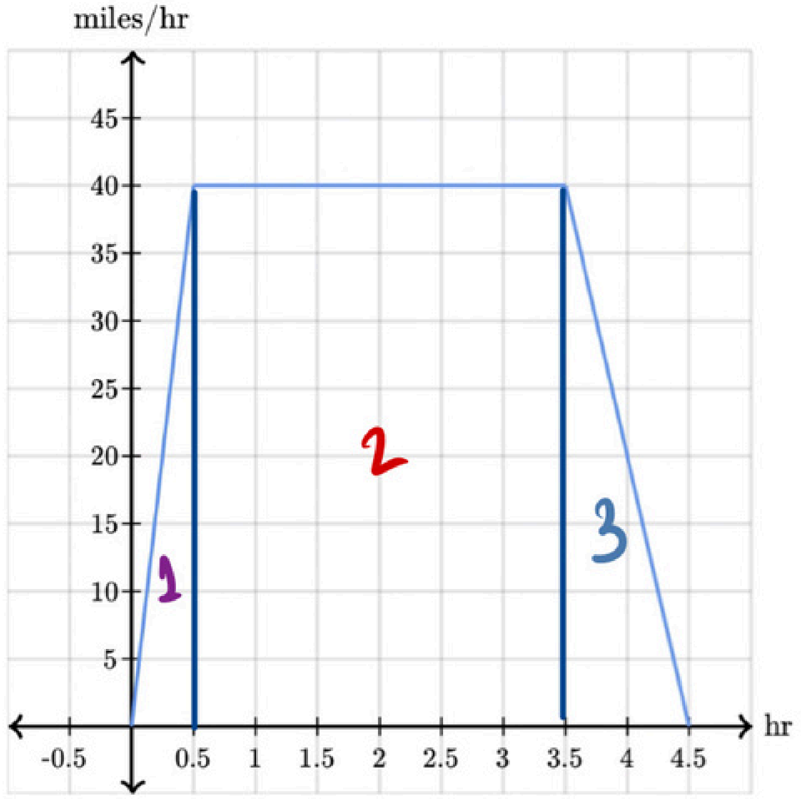
6.1 Exploring Accumulation of Change

A freight train leaves the station on a 4.5-hour trip. The graph below shows the train's velocity as a function of time.



How far does the train travel?

[ ] miles

① Split the area up into easy shapes and get areas

Shape 1 (triangle)

Area $= \frac{1}{2}bh$

$= \frac{1}{2}(\frac{1}{2})(40) = 10$

Shape 2 (rectangle)

Area $= bh$

$= (3)(40) = 120$

Shape 3 (triangle)

Area $= \frac{1}{2}bh$

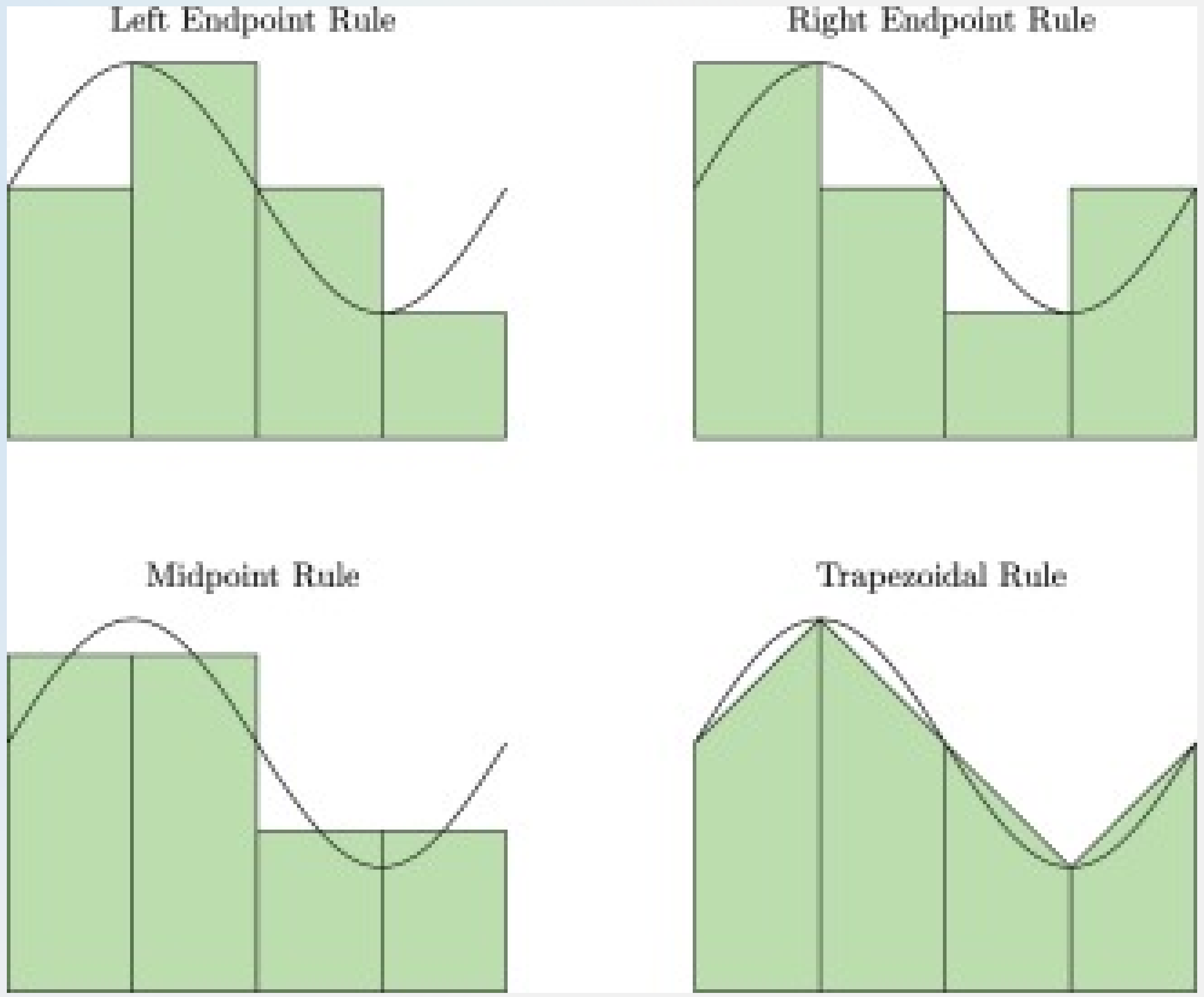
$= \frac{1}{2}(1)(40) = 20$

② Add all areas

$10 + 120 + 20 = \boxed{150 \text{ miles}}$

6.2 Approximating Areas with Riemann Sums

- Riemann sums: approximations of a definite integral using simple shapes
  - Left Riemann sums: rectangles touch the curve with their top-left corner
  - Right Riemann sums: rectangles touch the curve with their top-right corner
  - Midpoint Riemann sums: rectangles touch the curve with the point at the midpoint of its base
  - Trapezoidal Rule: Uses trapezoids to get more accurate dimensions

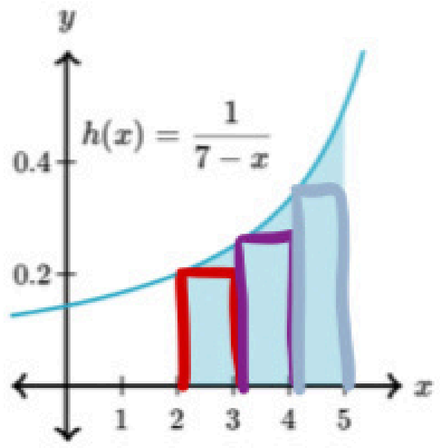


6.2 Approximating Areas with Riemann Sums

Approximate the area between the x-axis and $h(x) = \frac{1}{7-x}$ from $x = 2$ to $x = 5$ using a *left Riemann sum* with 3 equal subdivisions.

The approximate area is ☐ units^2.

Here's a sketch to help you visualize the area:



3 equal subdivisions between x=5 and x=2, each subdivision is $\frac{3}{5-2} = \frac{3}{3} = 1$ unit wide

Each rectangle's height is the value of f(x) at the left corner

$1\left(\frac{1}{7-2}\right) = 1\left(\frac{1}{5}\right) = \frac{1}{5}$ $\quad \frac{1}{5} + \frac{1}{4} + \frac{1}{3} = \boxed{\frac{47}{60}}$

$1\left(\frac{1}{7-3}\right) = 1\left(\frac{1}{4}\right) = \frac{1}{4}$

$1\left(\frac{1}{7-4}\right) = 1\left(\frac{1}{3}\right) = \frac{1}{3}$

6.2 Approximating Areas with Riemann Sums

Type of approximation	Increasing Function	Decreasing Function	Concave Up	Concave Down
Left Riemann	Underestimate	Overestimate	-	-
Right Riemann	Overestimate	Underestimate	-	-
Midpoint	-	-	Underestimate	Overestimate
Trapezoidal	-	-	Overestimate	Underestimate

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

- As the width of each rectangle in a Riemann sum, gets smaller and smaller, it gets closer to the actual value of the function (definite integral).
- A definite integral can be translated into the limit of a related Riemann sum, and vice versa!
- The summation notation form of a right Riemann sum is shown below. Using our definition of change in x, we get

$$\sum_{i=1}^{n} f(a + i\Delta x) \cdot \Delta x.$$

$$\Delta x = \frac{b-a}{n}$$

$$\sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}.$$

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

- As the width of each rectangle in a Riemann sum, gets smaller and smaller, it gets closer to the actual value of the function (definite integral).
- The formula for a right Riemann sum is shown below, followed by a left Riemann sum.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i \cdot \frac{b-a}{n}) \cdot \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(a + i \cdot \frac{b-a}{n}) \cdot \frac{b-a}{n}$$

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

Which of the definite integrals is equivalent to the following limit?

$$\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{\pi}{2}+\frac{\pi i}{2n}\right)\cdot\frac{\pi}{2n}$$

Choose 1 answer:

(A) $\int_0^{\pi}\cos x\,dx$

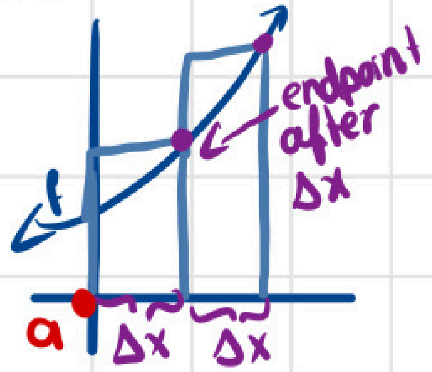
(B) $\int_0^{\pi/2}\cos x\,dx$

(C) $\int_{\pi/2}^{3\pi/4}\cos x\,dx$

(D) $\int_{\pi/2}^{\pi}\cos x\,dx$

The right Riemann sum is given by $\sum_{i=1}^{n} \underbrace{f(\underbrace{a}_{}+\underbrace{i\Delta x}_{})}_{} \underbrace{\Delta x}_{\text{width}}$

height of rectangle, a is leftmost bound and we start at i=1 since we want to calculate f at the right endpoint, as in the following:



for instance the first endpoint would be at i=1, $f(a+\Delta x)\Delta x$. This is also why we start at i=0 in left Riemann sums, since we don't want to move one Δx over.

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

Since our $\Delta x = \frac{b-a}{n}$ ← total distance between the endpoints ← divided by total subdivisions,

and our n (# of divisions) is growing increasingly larger as we chop up the rectangles in increasingly small slices to approximate the integral,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{(a-b)}{n}\right)\left(\frac{a-b}{n}\right)$$

We can match up parts of this formula from the given expression, $\lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{\pi}{2} + \frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}$

$f(x) = \cos(x)$ $\quad a = \frac{\pi}{2}$ $\quad \frac{i(a-b)}{n} = \frac{i\pi}{2n}$ $\quad \frac{a-b}{n} = \frac{\pi}{2n}$

simplify

$a - b = \frac{\pi}{2}$

substitute $a = \frac{\pi}{2}$

$\frac{\pi}{2} - b = \frac{\pi}{2}$

solve

$\pi = b$

Now we have everything we need to substitute into $\int_a^b f(x)\,dx$,

so

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \cos\left(\frac{\pi}{2} + \frac{i\pi}{2n}\right)\left(\frac{\pi}{2n}\right) = \int_{\frac{\pi}{2}}^{\pi} \cos(x)\,dx$$

6.4 The Fundamental Theorem of Calculus Accumulation Function

- As the width of each rectangle in a Riemann sum, gets smaller and smaller, it gets closer to the actual value of the function (definite integral).
- A definite integral can be translated into the limit of a related Riemann sum, and vice versa!

If f is a continuous function on an interval containing a, then $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, where x is in the interval.

Composite Functions and The Second Fundamental Theorem of Calculus

When the upper limit of the integral is a function of x rather than x itself:

$$A(x) = \int_a^{g(x)} f(t)\,dt$$

We can use the Second Fundamental Theorem of Calculus together with the Chain Rule to differentiate the integral:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

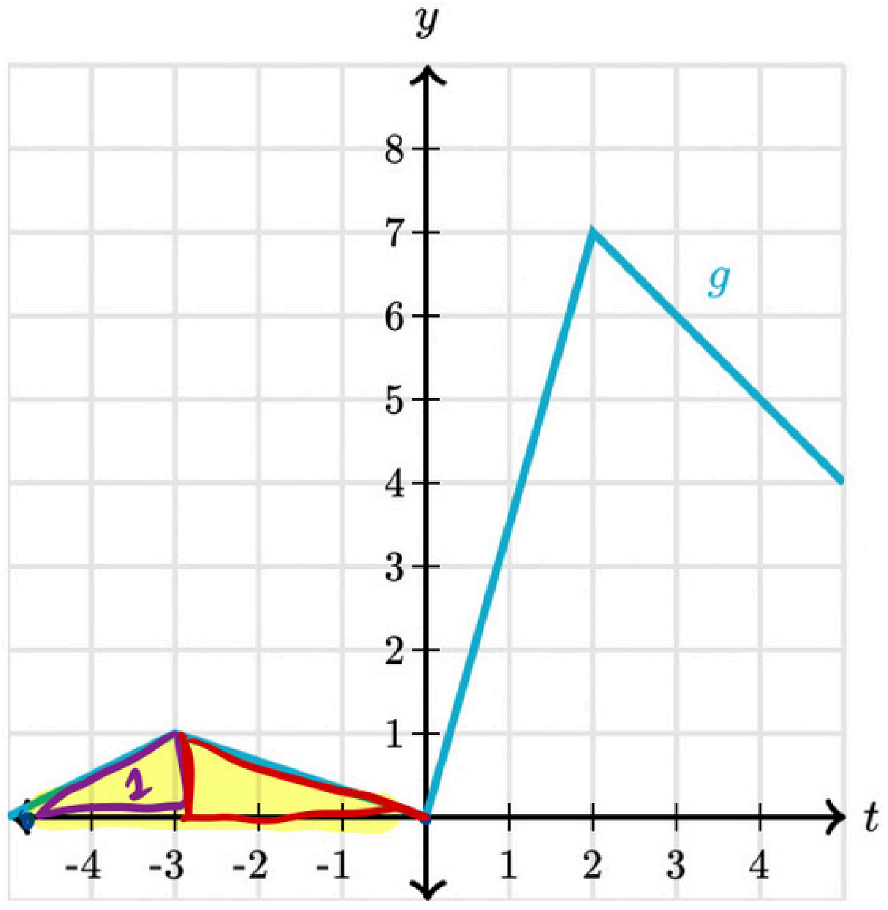
6.4 The Fundamental Theorem of Calculus Accumulation Function

The graph of function g is shown below. Let

$$h(x) = \int_{-5}^{x} g(t)\,dt.$$

① $h(0) = \int_{-5}^{0} g(t)\,dt$ Substitute

area under the curve from $x=-5$ to $x=0$



② Find area by splitting up into simpler shapes

Shape 1: triangle

Area $= \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$

Shape 2: triangle

Area $= \frac{1}{2}bh = \frac{1}{2}(3)(1) = \frac{3}{2}$

③ Add up simpler shapes

$1 + \frac{3}{2} = \boxed{\frac{5}{2}}$

Evaluate $h(0)$.

$$F(x) = \int_{0}^{\sqrt{x}} 2t\,dt$$

substitute variable bound

Don't forget chain rule w/ substituted bound

$F'(x) = 2\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x})$

$= 2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$

$\boxed{=1}$

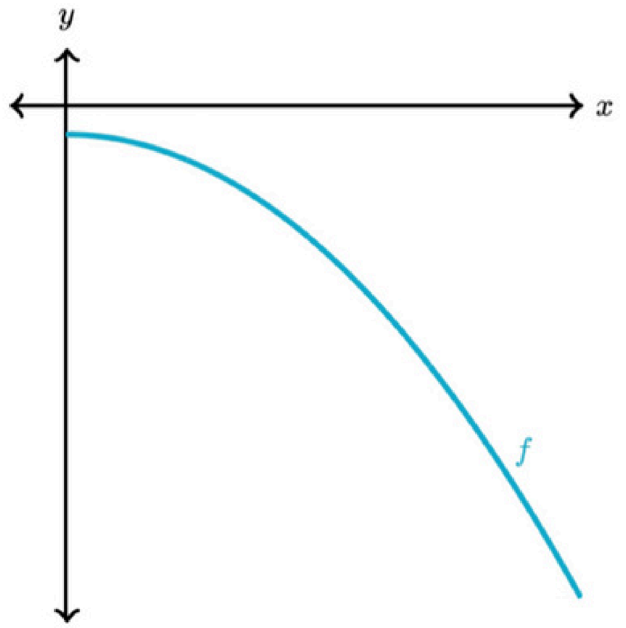
where $x > 0$.

$F'(x) =$ ☐

6.5 Interpreting the Behavior of Accumulation Functions Involving Area

- We can use the first and second derivatives of accumulation functions of the form $F(x) = \int_a^x f(t)\,dt$ to analyze a function's concavity, maximums/minimums, and points of inflection just like a normal function!
  - Use the FTC

6.5 Interpreting the Behavior of Accumulation Functions Involving Area



Let $g(x) = \int_0^x f(t)\,dt$.

make it related to derivatives or integrals

What is an appropriate calculus-based justification for the fact that g is decreasing?

← rate of change, derivative

$g'(x) = f(x)$ ← FTC

Since $f(x)$ is negative, $g'(x)$ is negative so g is decreasing.

6.6 Applying Properties of Definite Integrals

Sum/Difference: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Constant multiple: $\int_a^b k \cdot f(x)dx = k \int_a^b f(x)dx$

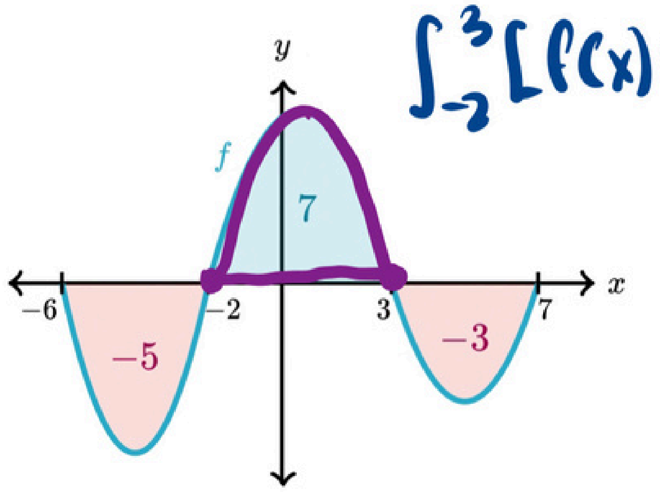
Reverse interval: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

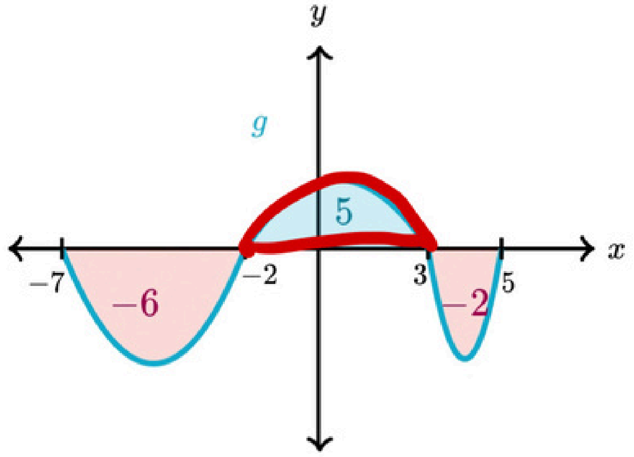
Zero-length interval: $\int_a^a f(x)dx = 0$

Adding intervals: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

6.6 Applying Properties of Definite Integrals

$\int_{-2}^{3}[f(x)+g(x)]\,dx = \square$



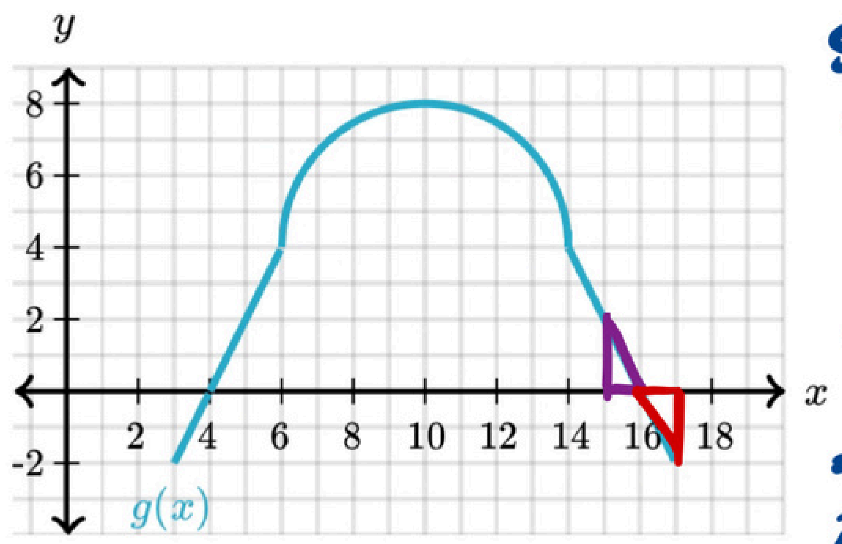


$\int_{-2}^{3}[f(x)+g(x)]dx = \underline{\int_{-2}^{3} f(x)dx} + \underline{\int_{-2}^{3} g(x)dx}$

$= 7 + 5$

$\boxed{= 12}$

The graph of g is comprised of a semi-circle and line segments.



① Split up into easier shapes

Triangle 1

$\frac{1}{2}(1)(2) = 1$

Triangle 2

$-\left(\frac{1}{2}(1)(2)\right) = -1$

negative since under the x-axis

② Add up

$1 - 1 = \boxed{0}$

Evaluate the definite integral of $\int_{15}^{17} g(x)\,dx$.

6.7 The Fundamental Theorem of Calculus and Definite Integrals

- An integral is the **anti**derivative of the function
  - An antiderivative of a function f(x) is a function F(x) whose derivative is f(x)
- If a function f is continuous on an interval containing a, the function defined by $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f for x in the interval.

First Fundamental Theorem of Calculus

Given *f* is

- continuous on interval [a, b]
- *F* is any function that satisfies *F'*(x) = *f*(x)

Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Power Rule

Derivatives

1) ***Multiply*** by exponent
2) ***Subtract*** 1 from exponent

$y = x^n$

$\frac{dy}{dx} = nx^{n-1}$

Integrals

1) ***Add*** 1 to exponent
2) ***Divide*** by new exponent

$y = x^n$

$$\boxed{\int y\,dx = \frac{x^{n+1}}{n+1} + C}$$

6.7 The Fundamental Theorem of Calculus and Definite Integrals

$F(x) = \sqrt{x+7}$

$f(x) = F'(x)$

$\int_2^9 f(x)\,dx = \square$

$$\int_2^9 f(x)\,dx = \int_2^9 F'(x)\,dx = F(x)\Big|_2^9 \quad \text{by FTC}$$

$$= F(9) - F(2)$$

$$= \sqrt{9+7} - \sqrt{2+7}$$

$$= \sqrt{16} - \sqrt{9} = 4 - 3 = 1$$

6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

- If you are not given bounds (as in an antiderivative), you must include a +C to represent a constant
- Remember how constants go away when we take a derivative? By adding +C, we're accounting for the constant that could've been taken away.
- Here are some formulas for antiderivatives of trig functions:
  - Remember: An antiderivative of a function f(x) is a function F(x) whose derivative is f(x)
- You can check if the antiderivative is right by differentiating it and seeing if it matches what you had at first!

Derivatives or Differentiation Formulas	Antiderivatives or Integration Formulas
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$

6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

Reverse power rule

$$\int \sqrt[3]{x^7}\,dx = \square + C$$

$$\int x^{\frac{7}{3}} = \frac{x^{(\frac{7}{3}+1)}}{(\frac{7}{3}+1)} + C = \frac{x^{\frac{10}{3}}}{\frac{10}{3}} + C$$

$$\boxed{= \frac{3}{10}x^{\frac{10}{3}} + C}$$

Integrate.

$$\int \left(-\frac{3}{x} + 3e^x\right) dx = ?$$

$$= \int -\frac{3}{x}\,dx + \int 3e^x\,dx$$

$$\boxed{= -3\ln|x| + 3e^x + C}$$

Integrate.

$$\int 6\sin(x)\,dx$$

$$= 6\int \sin(x)\,dx$$

$$\boxed{= -6\cos(x) + C}$$

6.9 Integrating Using Substitution

- U-substitution in integration is similar to the chain rule in differentiation (it is sort of like the reverse!)

cos is within another function (x^3)

$$\int_0^{\pi/3} \sin(x)\cos^3(x)\,dx = \int_1^{\frac{1}{2}} -u^3\,du = -\int_{\frac{1}{2}}^{1} -u^3\,du$$

$$= \int_{\frac{1}{2}}^{1} u^3\,du = \frac{u^4}{4}\Big|_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4} - \frac{\left(\frac{1}{2}\right)^4}{4} = \frac{1}{4} - \frac{\frac{1}{16}}{4}$$

$$= \frac{1}{4} - \frac{1}{64} = \boxed{\frac{15}{64}}$$

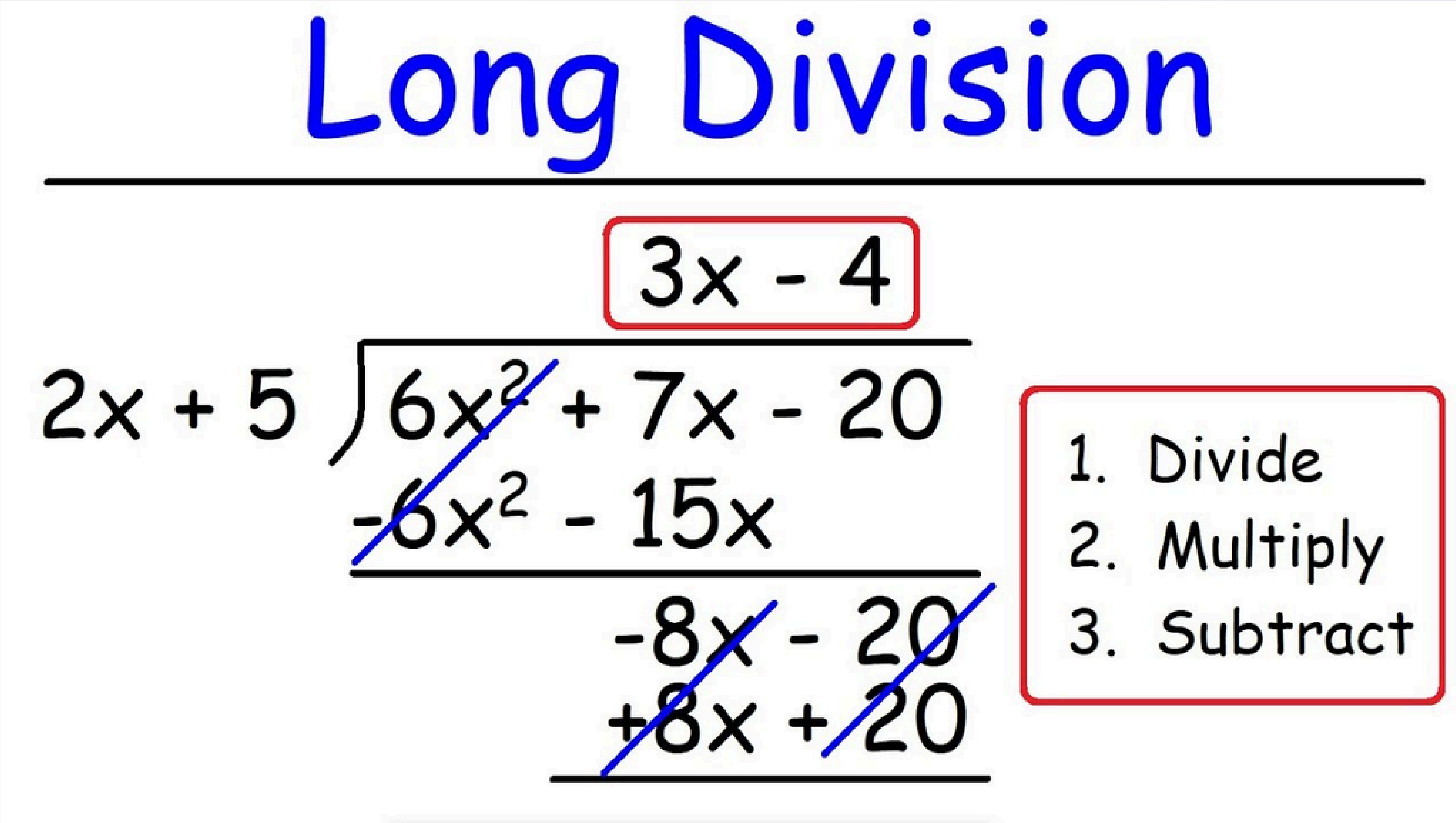
$u = \cos(x)$

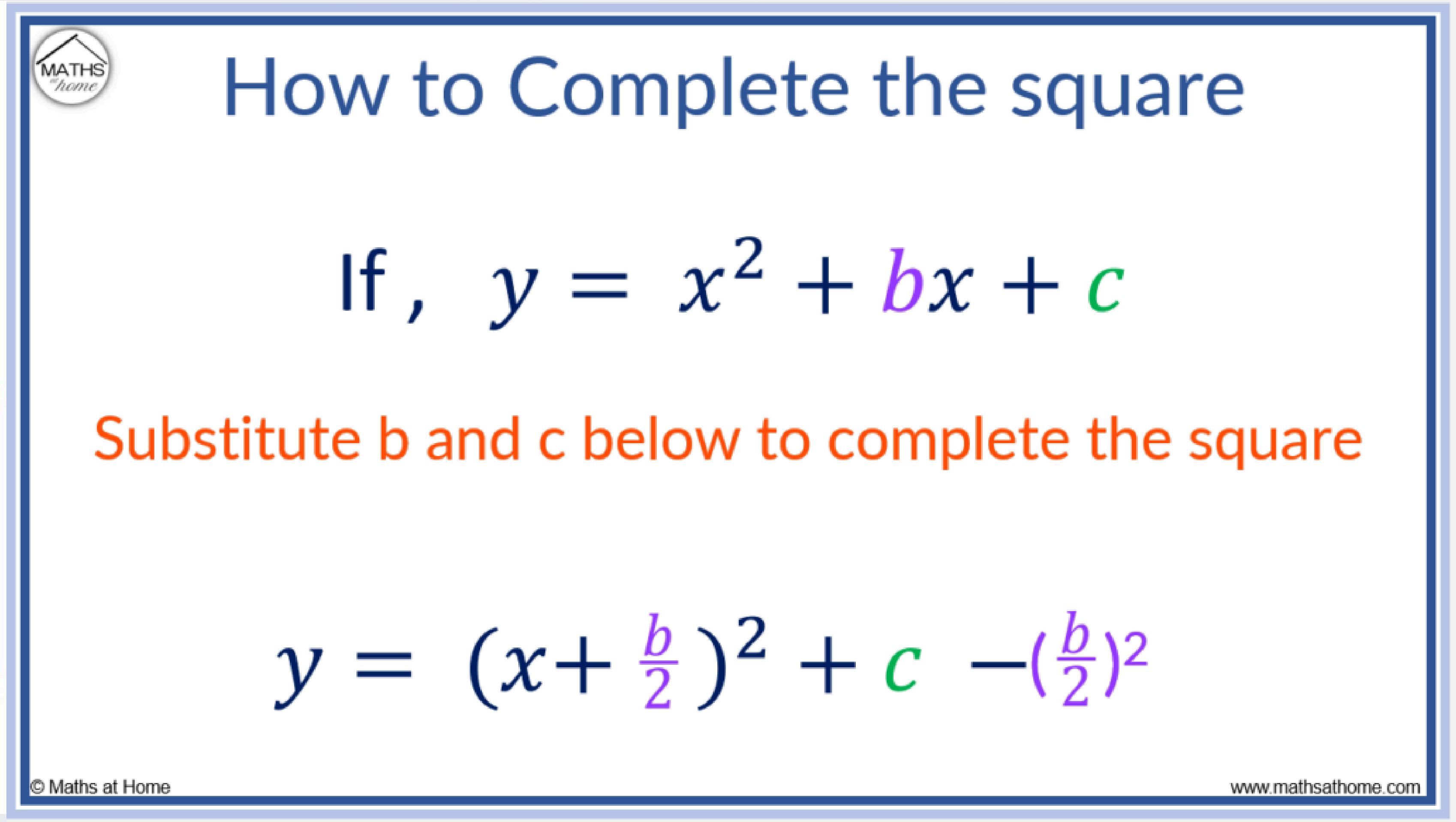
$du = -\sin(x)$

endpoints: $u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$u = \cos(0) = 1$

6.10 Integrating Functions Using Long Division and Completing the Square





Rearrange the polynomials into equivalent forms to make them easier to integrate!!!

6.10 Integrating Functions Using Long Division and Completing the Square

Evaluate $\int \frac{2x^3+7x^2+2x+9}{2x+3}\,dx$.

what can 2x be multiplied by to get the # inside?

$$
\begin{array}{r}
x^2+2x-2 \\
2x+3\,\overline{)\,2x^3+7x^2+2x+9} \\
-(2x^3+3x^2) \downarrow \quad \downarrow \\
\hline
4x^2+2x+9 \\
-(4x^2+6x) \\
\hline
-4x+9 \\
-(-4x-6) \\
\hline
15
\end{array}
$$

result of the division

remainder

$$x^2+2x-2+\frac{15}{2x+3}$$

$$\int \frac{2x^3+7x^2+2x+9}{2x+3}\,dx = \int x^2+2x-2+\frac{15}{2x+3}\,dx$$

$$\boxed{=\frac{x^3}{3}+x^2-2x+\frac{15}{2}\ln|2x+3|+C}$$

6.11 Integrating Using Integration by Parts (BC ONLY)

Antiderivative

(indefinite integral)

$$\int u\,dv = uv - \int v\,du$$

Definite integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

6.12 Integrating Using Linear Partial Fraction Decomposition (BC ONLY)

1. Factor the denominator into linear factors (highest power of x has to be 1)

Evaluate $\int \frac{2x^2-7}{x^3-3x^2-4x}\,dx$ $\rightarrow$ $x^3-3x^2-4x = x(x^2-3x-4) = x(x-4)(x+1)$

2. Make each factor the denominator to a new fraction with A, B, C ... so on as placeholders in the numerator. Add all these new fractions up and equal them to the old fraction as in this example:

$$\frac{2x^2-7}{x^3-3x^2-4x} = \frac{2x^2-7}{x(x-4)(x+1)} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+1}$$

6.12 Integrating Using Linear Partial Fraction Decomposition (BC ONLY)

3. Multiply both sides by all the factors of the denominator of the original fraction. This should get rid of the denominator of the original fraction.

$$\left(x(x-4)(x+1)\right)\left(\frac{2x^2-7}{x(x-4)(x+1)}\right)=\left(\frac{A}{x}+\frac{B}{x-4}+\frac{C}{x+1}\right)\left(x(x-4)(x+1)\right)$$

$$2x^2-7=A(x-4)(x+1)+B(x)(x+1)+C(x)(x-4)$$

4. Equal each of the linear factors to zero and then plug in each x value. This should give you the value of each placeholder (continued on next page).

$$x-4=0 \qquad x+1=0 \qquad x=0$$

$$x=4 \qquad x=-1$$

6.12 Integrating Using Linear Partial Fraction Decomposition (BC ONLY)

Let $x=4$, $2(4)^2-7 = A(4-4)(4+1) + \underline{B(4)(4+1) + C(4)(4-4)}$

$2(16)-7 = B(4)(5)$ $\quad 20B = 25$

$32-7 = 20B$ $\quad B = \frac{5}{4}$

Let $x=-1$, $2(-1)^2-7 = A(-1-4)(-1+1) + B(-1)(-1+1) + C(-1)(-1-4)$

$2-7 = 5C$

$C = -1$

Let $x=0$ $\quad 2(0)^2-7 = A(0-4)(0+1) + B(0)(0+1) + C(0)(0-4)$

$-7 = -4A$

$A = \frac{7}{4}$

6.12 Integrating Using Linear Partial Fraction Decomposition (BC ONLY)

5. Integrate the newly decomposed fractions. Since this decomposed form is equal to the original, it gives the same result as integrating the OG!

$$\int \left(\frac{2x^2-7}{x^3-3x^2-4x}\right)dx = \int \left(\frac{\frac{7}{4}}{x} + \frac{\frac{5}{4}}{x-4} + \frac{-1}{x+1}\right)dx$$

$$= \int \frac{7}{4}\left(\frac{1}{x}\right) + \frac{5}{4}\left(\frac{1}{x-4}\right) - \left(\frac{1}{x+1}\right) dx$$

$$\boxed{= \frac{7}{4}\ln|x| + \frac{5}{4}\ln|x-4| - \ln|x+1| + C}$$

6.13 Evaluating Improper Integrals (BC ONLY)

Improper Integrals

$$1.\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$2.\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

$$3.\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

If the limit exists then the improper integral converges.
If the limit does not exists then the improper integral diverges.

Examples:

$$\int_0^{\infty} 4e^{-2x}dx = \lim_{b\to\infty}\int_0^b 4e^{-2x}dx$$
$$= \lim_{b\to\infty}\left[-2e^{-2x}\right]_0^b$$
$$= \lim_{b\to\infty}\left[-2e^{-2b} - \left(-2e^{0}\right)\right]$$
$$= 2$$ (converges)

$$\int_1^{\infty} \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx$$
$$= \lim_{b\to\infty}\left[\ln x\right]_1^b$$
$$= \lim_{b\to\infty}\left[\ln b - \ln 1\right]$$
(diverges)

6.13 Evaluating Improper Integrals

(BC ONLY)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

If we tried to plug in $x=0$, the answer is undefined. We use limits to deal with this.

$$\lim_{t\to 0} \int_t^1 \frac{1}{\sqrt{x}} = \lim_{t\to 0} \left(2\sqrt{x}\,\Big|_t^1\right)$$

$$= \lim_{t\to 0} (2\sqrt{1} - 2\sqrt{t})$$

$$= 2\sqrt{1} - 2(0)$$ ← direct substitution

Thank You

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