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UNIT 4

Contextual Applications of Differentiation

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4.1 INTERPRETING THE MEANING OF THE DERIVATIVE IN CONTEXT

- The derivative of a function is just the rate of change at that point
- The unit for $f'(x)$, or the rate of change of $f(x)$, is just the unit of $f(x)$ divided by the unit for x.
    - For example, if we have $f(x)$ in grams and x in seconds the unit for rate of change $f'(x)$ would be grams/second.

4.1 INTERPRETING THE MEANING OF THE DERIVATIVE IN CONTEXT

P gives the population of a town t years after it was founded.

What is the best interpretation for the following statement?

The slope of the line tangent to the graph of P at $t = 2$ is equal to -10.

Choose 1 answer:

- (A) After 2 years, the rate of change of the population decreased at a rate of 10 people per year per year.
- (B) After 2 years, the town's population decreased at a rate of 10 people per year.
- (C) Over the first two years, the town's population decreased by 10 people per year.
- (D) After 2 years, the town's population decreased at a rate of 10 people.

P is in units of people. The derivative of P will give $\frac{\text{People}}{\text{year}}$, or people per year, since it tells the rate of change of people in the town per year (unit given in the problem). Slope of the tangent line is the same as the derivative, or rate of change at that point, so after 2 years (at $t=2$) the rate of the town's population is decreasing (since slope is negative) at a rate of of 10 people per year.

4.2 STRAIGHT LINE MOTION: POSITION, VELOCITY, ACCELERATION

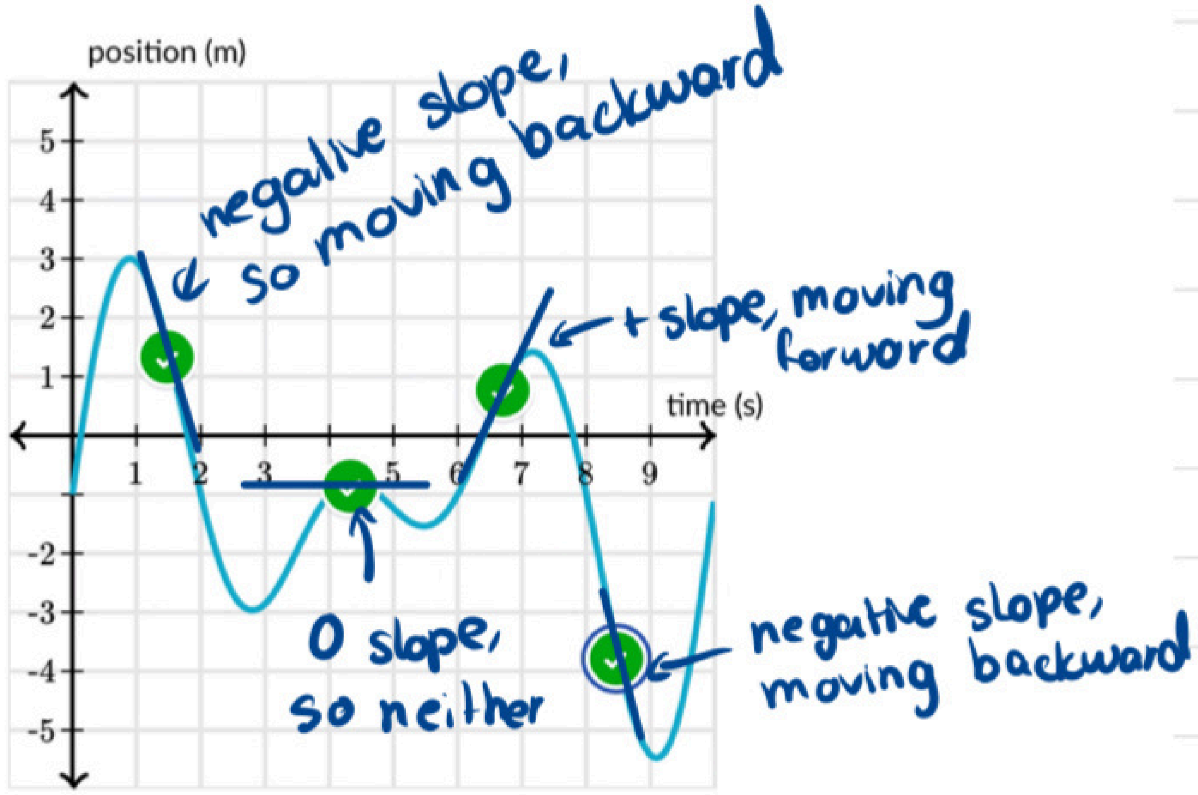
- Velocity is the derivative of position
- Acceleration is the derivative of velocity
- If velocity and acceleration have the same sign, the particle is speeding up
- If velocity and acceleration have different signs, the particle is slowing down.
- If velocity is negative, the particle is moving left. If velocity is positive, it is moving to the right. If velocity is 0 the particle is not moving.

4.2 STRAIGHT LINE MOTION: POSITION, VELOCITY, ACCELERATION

An object is moving along a line. The following graph gives the object's position, relative to its starting point, over time.

For each point on the graph, is the object moving forward, backward, or neither?

Tap each dot on the image to select an answer.



The object is moving forward if velocity (derivative of velocity) is +. Moving backward if velocity is negative.

Since velocity is the derivative of position, we look at the tangent line at each point to determine if it is + or −.

You can also picture this as the object moving forward or backward to $y=0$.

4.2 STRAIGHT LINE MOTION: POSITION, VELOCITY, ACCELERATION

A particle moves along the x-axis. The function $x(t)$ gives the particle's position at any time $t \geq 0$

$$x(t) = t^3 - 3t^2 + 7t - 6$$

What is the particle's acceleration $a(t)$ at $t = 3$?

$a(3) =$ ☐

Since acceleration is the second derivative of position,

$a(t) = v'(t) = x''(t)$

$a(t) = (t^3 - 3t^2 + 7t - 6)'' = (3t^2 - 6t + 7)' = 6t - 6$

$a(3) = 6(3) - 6 = \boxed{12}$

4.3 RATES OF CHANGE IN OTHER CONTEXTS

- If something is increasing, its derivative (rate of change) is positive.
- If something is decreasing, its derivative (rate of change) is negative.
- Finding derivatives on your calculator is helpful
- Be careful of functions that are already rates of change. You don't need to take the derivative of those since they already told you the derivative.
- The second derivative of a function is just the rate at which the rate of change is increasing
    - Represented by (units of f(x))/(units of x)^2
    - For example, grams/(sec^2)

4.3 RATES OF CHANGE IN OTHER CONTEXTS

Carbon-14 is an element which loses half of its mass every 5730 years. The following function gives the mass, in grams, of a sample of carbon-14 after t years:

$$M(t) = 65 \cdot e^{-0.00012t}$$

What is the instantaneous rate of change of sample's mass after 1 year?

Instantaneous rate of change is $M'(t)$

$$M'(t) = 65(-0.00012)e^{-0.00012t}$$

$$= -0.0078e^{-0.00012t}$$

$$M'(1) = -0.0078e^{-0.00012} \leftarrow \text{at } t=1$$

$$\boxed{= -0.0078 \ \frac{\text{grams}}{\text{year}}}$$

4.4 INTRODUCTION TO RELATED RATES

- If variables are related to each other in an equation, we can find an equation that relates the rate of change of the variables.
    - Done through implicit differentiation
- We want to find the rate at which one variable is changing by relating it to other variables whose rates of change are known.

4.4 INTRODUCTION TO RELATED RATES

Consider the following problem:

> The radius $r(t)$ of the base of a cylinder is decreasing at a rate of 9 millimeters per hour and the height $h(t)$ of the cylinder is increasing at a rate of 2 millimeters per hour. At a certain instant t_0, the base radius is 8 millimeters and the height is 3 millimeters. What is the rate of change of the surface area $S(t)$ of the cylinder at that instant?

Match each expression with its given value.

	-9	2	3	8	Not given
$S(t_0)$	○	○	○	○	●
$r'(t_0)$	●	○	○	○	○
$h'(t)$	○	●	○	○	○

① First, we go through the problem and list what they give us.

$r'(t) = -9$ ← decreasing, so negative

$r'(t)$ since it gives us the rate of change, not the actual radius

$h'(t) = 2$ ← increasing, so +

$h'(t)$ since it gives rate of change

$r(t_0) = 8$ $\quad$ $h(t_0) = 3$

gives us actual radius & base at t_0

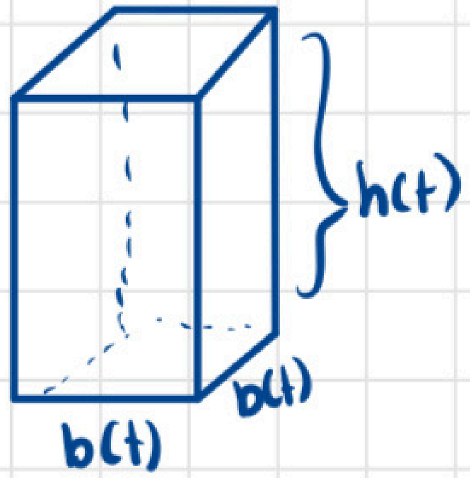
② Referencing our list of values above, $S(t_0)$ is not given.
$r'(t_0)$ is given since $r'(t) = -9$, which means that at any time t the rate of change is -9, including at t_0.

$h'(t)$ is given as $h'(t) = 2$.

4.4 INTRODUCTION TO RELATED RATES

The side $b(t)$ of the base of a square prism is decreasing at a rate of 7 kilometers per minute and the height $h(t)$ of the prism is increasing at a rate of 10 kilometers per minute. At a certain instant t_0, the base's side is 4 kilometers and the height is 9 kilometers. What is the rate of change of the surface area $S(t)$ of the prism at that instant?

Which equation should Urpi use to solve the problem?



① Get all info

$b'(t) = -7$ $\quad$ $h'(t) = 10$

$b(t_0) = 4$ $\quad$ $h(t_0) = 9$

$S'(t) = ?$

② We want $S'(t)$. Let's think of an equation that gives surface area of a square prism:

$$S(t) = \underbrace{2}_{\text{accounting for the top and bottom}}\underbrace{[b(t)]^2}_{\text{area of square side of the prism}} + \underbrace{4}_{\text{accounting for all 4 sides}}\underbrace{b(t)\cdot h(t)}_{\text{area of side}}$$

Urpi should use the equation $S(t)$ above.

4.4 INTRODUCTION TO RELATED RATES

The differentiable functions x and y are related by the following equation:

$$3x^2 = xy$$

Also, $\frac{dy}{dt} = -3$.

Find $\frac{dx}{dt}$ when $x = -6$.

① We can differentiate w/ respect to t

$$\frac{d}{dt}(3x^2) = \frac{d}{dt}(xy)$$

$$6x\frac{dx}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

② Plug in $x=-6$, $\frac{dy}{dt}=-3$ and solve for $\frac{dx}{dt}$ as you would any variable.

$$6(-6)\frac{dx}{dt} = \frac{dx}{dt}y - 6(-3)$$

But wait! We don't know what y is :(. Don't worry, we can use the o.g. equation w/ $x=-6$ to figure it out.

$$3(-6)^2 = (-6)y \qquad 3(36) = -6y \qquad -3(6) = y = -18$$

Now that we have $y=-18$, solve!

$$6(-6)\frac{dx}{dt} = \frac{dx}{dt}(-18) - 6(-3)$$

$$-36\frac{dx}{dt} = -18\frac{dx}{dt} + 18$$

$$-18\frac{dx}{dt} = 18$$

$$\boxed{\frac{dx}{dx} = -1}$$

4.5 SOLVING RELATED RATES PROBLEMS

To solve a related rates problem, we first have to make an equation that relates the variables. (e.g. pythagorean theorem to relate sides of a right triangle). We then take the derivative to find the rates of change. Next, substitute for what you know and you' re done! You might have to look at the o.g. equation and plug in variables (e.g. plug in x to find y) to get all the numbers you need to be able to solve.

4.5 SOLVING RELATED RATES PROBLEMS

The base of a triangle is *decreasing* at a rate of 13 millimeters per minute and the height of the triangle is increasing at a rate of 6 millimeters per minute.

At a certain instant, the base is 5 millimeters and the height is 1 millimeter.

What is the rate of change of the area of the triangle at that instant (in square millimeters per minute)?

① Get your variables

$b'(t) = -13 \qquad h'(t) = 6 \qquad b(t_0) = 5 \qquad h(t_0) = 1$

② Formula for area of triangle

Area $= \frac{1}{2}$ base · height, so

$A(t) = \frac{1}{2} b(t) \cdot h(t)$

③ Differentiate $A(t)$ to find rate of change of area

$A'(t) = \left(b'(t)h(t) + b(t)h'(t)\right) \cdot \frac{1}{2}$

$A'(t_0) = \left(b'(t_0)h(t_0) + b(t_0)h'(t_0)\right) \cdot \frac{1}{2}$ at instant t_0

$= \left((-13)(1) + (5)(6)\right) \cdot \frac{1}{2} = (+17) \cdot \frac{1}{2} = \boxed{+8.5 \ \frac{mm^2}{min}}$

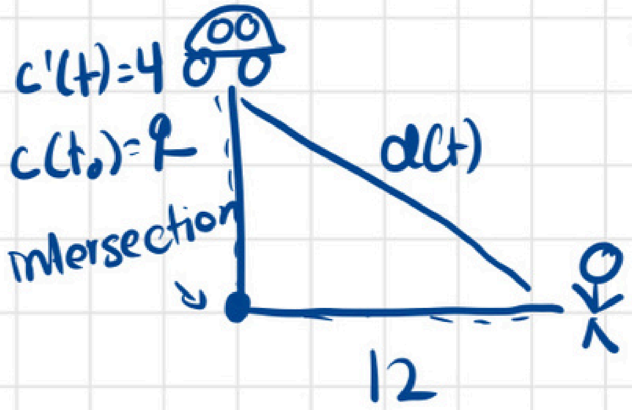
4.5 SOLVING RELATED RATES PROBLEMS

A person stands 12 meters east of an intersection and watches a car driving *away from* the intersection to the north at 4 meters per second.

At a certain instant, the car is 9 meters from the intersection.

What is the rate of change of the distance between the car and the person at that instant (in meters per second)?

① Draw a diagram to make sense of the situation:



We can see that this is a right triangle. $c'(t)=4$ represents the rate at which the distance increases. The distance at the certain time $t_0 = c(t_0) = 4$

$d(t)$ is the distance between the person and car.

② Use the pythagorean theorem to relate $d(t)$ to what we know. Find $d(t_0)$.

$[d(t)]^2 = [c(t)]^2 + 12^2 \quad [d(t_0)]^2 = [c(t_0)]^2 + 144 \Rightarrow [d(t_0)]^2 = 81 + 144 = 225 \Rightarrow \underline{d(t_0) = 15}$

③ Take the derivative to find $d'(t)$

$$2d(t) \cdot d'(t) = 2c'(t) \cdot c(t)$$

④ Find the derivative at $t = t_0$, $2d(t_0)d'(t_0) = 2c(t_0)c'(t_0)$ and substitute,

$$2(15)(d'(t_0)) = 2(9)(4)$$

$$(15)(d'(t_0)) = (9)(4)$$

$$d'(t_0) = \frac{36}{15} = \boxed{\frac{12}{5}}$$

4.6 APPROXIMATING VALUES OF A FUNCTION USING LOCAL LINEARITY AND LINEARIZATION

- We can use the tangent line of a function at a point to approximate values of the function at other points (use point slope form with f'(x) = slope)
- Tangent line approximation is an overestimate if the curve is concave down
- Underestimate if concave up
- Green lines below are tangent line approximations!

4.6 APPROXIMATING VALUES OF A FUNCTION USING LOCAL LINEARITY AND LINEARIZATION

Let g be a differentiable function with $g(-1) = 5$ and $g'(-1) = 2$.

What is the value of the approximation of $g(-0.9)$ using the function's local linear approximation at $x = -1$?

① Determine the equation of the tangent line at t=-1
Using point-slope form, we have point (-1,5) and slope 2 at that point, so

$$y - y_1 = m(x - x_1) \qquad y - 5 = 2(x - (-1)) \qquad y - 5 = 2(x+1)$$

plug in

② Approximate g(-0.9) using the tangent line.

$$y - 5 = 2(-0.9+1) \Rightarrow y - 5 = 2(0.1) \Rightarrow y = 0.2 + 5 = \boxed{5.2}$$

4.7 USING L'HOPITAL'S RULE FOR DETERMINING LIMITS OF INDETERMINATE FORMS

- Indeterminate form is when the ratio of two functions tends to 0/0 or infinity/infinity in the limit
- You can use L'Hopital's on Limits of indeterminate forms 0/0 or infinity/infinity!

L'Hopital's Rule

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If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} =$ indeterminate form

then :

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

4.7 USING L'HOPITAL'S RULE FOR DETERMINING LIMITS OF INDETERMINATE FORMS

Find $\lim\limits_{x\to 0} \dfrac{\tan(x)}{3x+\tan(x)}$.

Let $\dfrac{\tan(x)}{3x+\tan(x)} = \dfrac{f(x)}{g(x)}$. Since $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \tan(x) = \tan(0) = 0$ and $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} (3x+\tan(x)) = 0$, the limit $\lim\limits_{x\to 0} \dfrac{\tan(x)}{3x+\tan(x)}$ is in indeterminate form. Therefore, L'Hopital's rule may be used.

$$\lim_{x\to 0} \frac{\tan(x)}{3x+\tan(x)} = \lim_{x\to 0} \frac{\frac{d}{dx}(\tan(x))}{\frac{d}{dx}(3x+\tan(x))}$$

$$= \lim_{x\to 0} \frac{\sec^2(x)}{3+\sec^2(x)} = \boxed{\frac{1}{4}}$$

THANK YOU

I hope you can get helpful knowledge from this presentation. Good luck!

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