AP Calc Unit 5

Study Guide by Loops of Kindness

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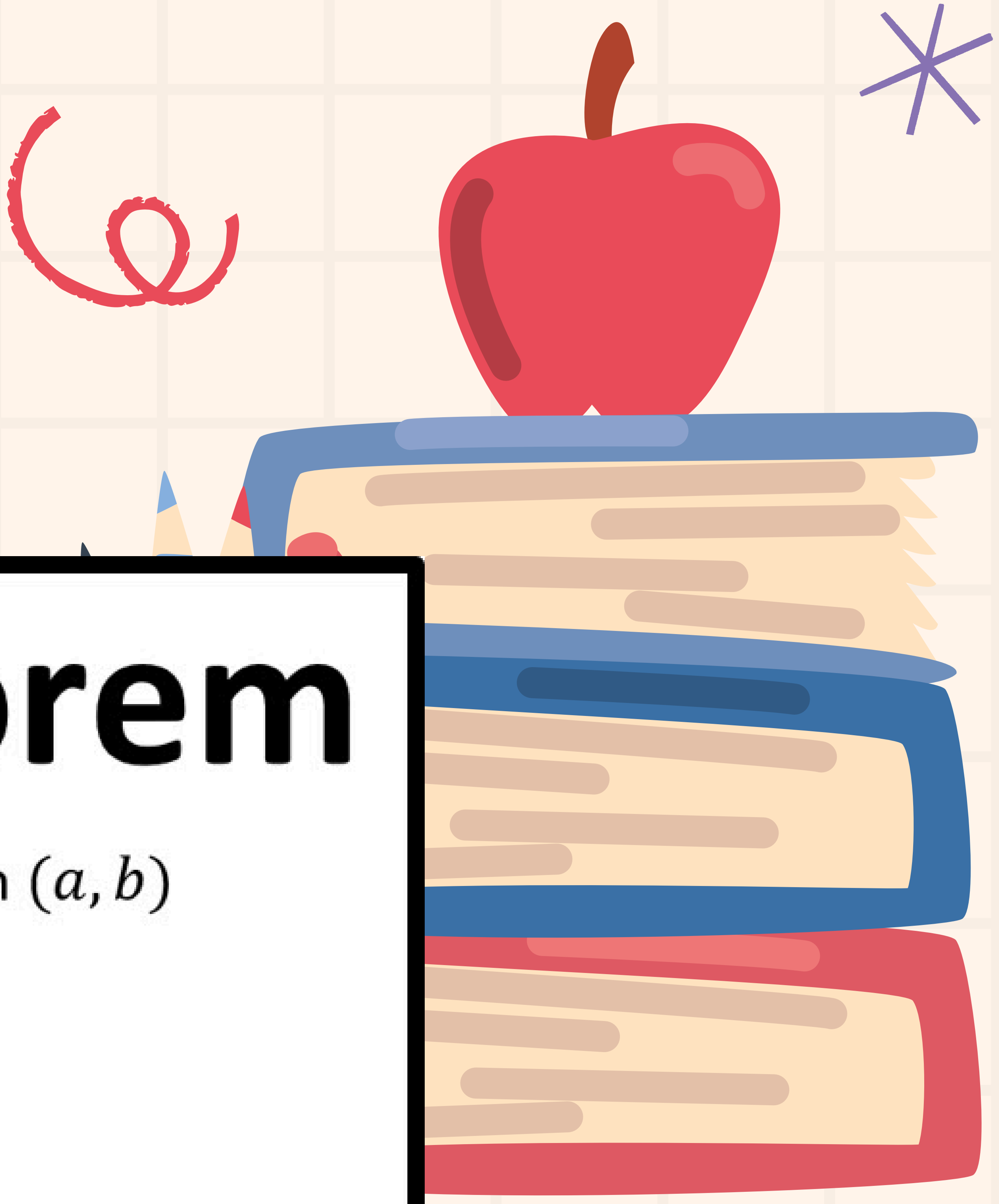
5.1 Mean Value Theorem



Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b)
then there is a c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

5.1 Mean Value Theorem

Let $f(x) = x^3 - 6x^2 + 12x$ and let c be the number that satisfies the Mean Value Theorem for f on the interval $[0, 3]$.

What is c ?

The MVT states that for a function continuous on $[a,b]$ and differentiable on (a,b), there exists a value c such that

$$a < c < b$$

and

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

← slope from a to b ☺

① Since the polynomial f(x) is continuous & differentiable on [0,3], the MVT applies. We can find $\frac{f(b)-f(a)}{b-a}$ and $f'(c)$ and equal them to each other to find c. This sounds semi confusing but you'll see how easy it is!

② $[a,b] = [0,3]$, $a=0$ $b=3$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(0)}{3-0} = \frac{27-6(9)+12(3)-0}{3} = \frac{27-54+36}{3} = \frac{9}{3} = 3$$

③ $f'(x) = 3x^2 - 12x + 12$, $f'(c) = 3c^2 - 12c + 12$

④ $f'(c) = \frac{f(b)-f(a)}{b-a}$ → solve for c

$3c^2 - 12c + 12 = 3 \Rightarrow 3c^2 - 12c + 9 = 0 \Rightarrow c^2 - 4c + 3 = 0$

$(c-3)(c-1) = 0$

$c = 3, 1$

Since $a < c < b$, $0 < c < 3$, $c \neq 3$ so $\boxed{c=1}$ is the only answer.

5.1 Mean Value Theorem

Let $h(x) = \log_2(x)$.

Can we use the mean value theorem to say the equation $h'(x) = 1$ has a solution where $1 < x < 2$?

$h'(x) = 1$ has a solution $x = c$ for $a < c < b$ by the MVT if

1. $h(x)$ is continuous on $[a,b] = [1,2]$
2. $h(x)$ is differentiable on $(a,b) = (1,2)$
3. $\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1} = 1$ $\left(h'(c) = \frac{f(b)-f(a)}{b-a} = 1\right)$

① $\log_2(x)$ is defined for $[1,2]$, so this condition is met.

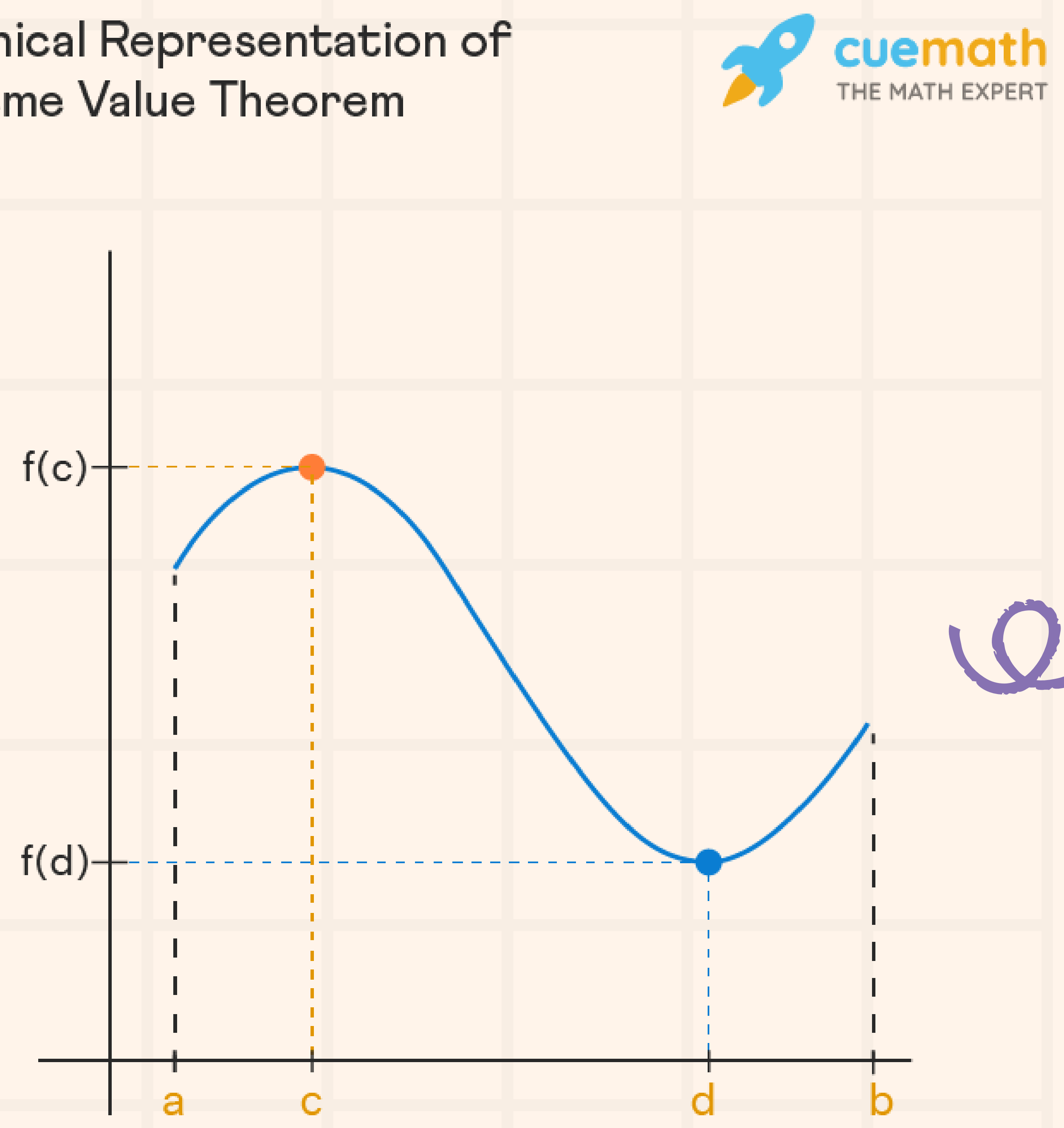
② Since $\log_2(x)$ is defined and smooth on $(1,2)$, this condition is met.

③ $\frac{f(2)-f(1)}{2-1} = \frac{\log_2(2) - \log_2(1)}{2-1} = \frac{1-0}{1} = 1$, so this condition is met.

Therefore, we can use the MVT to say the equation $h'(x) = 1$ has a solution where $1 < x < 2$.

5.2 Extreme Value Theorem

Graphical Representation of Extreme Value Theorem



If a function f is continuous over the interval [a, b], then f has at least one minimum value and at least one maximum value on [a,b].

It is essentially saying that a function continuous on an interval must have one value that's the highest (max) and one that's the lowest (min).

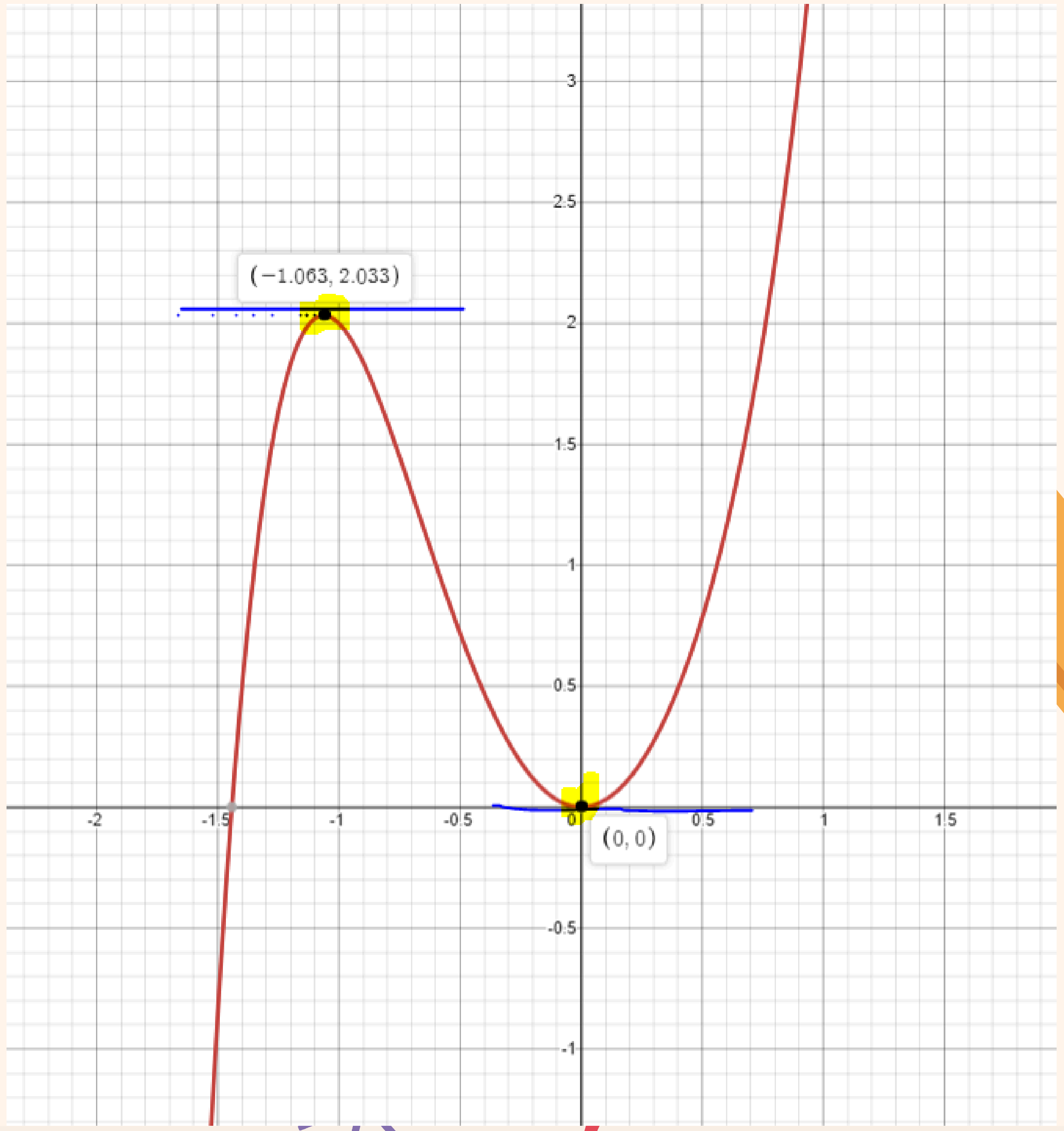
5.2 Critical Points

The critical points of a function is where the first derivative = 0 or is undefined.

All local (relative) extrema, or extrema that are not the endpoints, occur at critical points.

Remember: not all critical points are min/max, but all min/max happen at critical points.

To find critical points, just set f′(x) = 0



5.2 Critical Points

Let $f(x) = \dfrac{1}{x^2 - 4}$

Where does f have critical points?

f has critical points where its first derivative is 0 or undefined. The function f must also be defined at those points.

$f(x) = (x^2-4)^{-1}$ $f'(x) = -\dfrac{2x}{(x^2-4)^2}$

numerator

denominator

$f'(x) = 0$ at $-2x = 0$ $x = 0$ and is undefined at $(x^2-4)^2 = 0$ $x^2-4=0$ $x^2 = 4$ $x = \pm 2$. However, since the o.g. f is undefined at $x = \pm 2$, the only critical point is $x = 0$.

5.3 Increasing or Decreasing?

A function is increasing when the first derivative is positive, and decreasing when the first derivative is negative.

Remember that derivative is just the slope of the tangent line at a point. If the slope of a function at a point is negative the function is decreasing. If the slope at a point is positive, then the function is increasing.

5.3 Increasing or Decreasing?

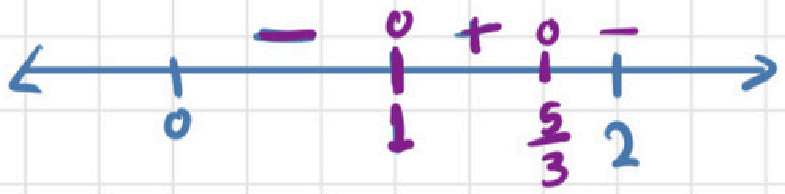
Let $g(x) = -x^3 + 4x^2 - 5x$.

On which intervals is g increasing?

① Find critical points: $g'(x) = -3x^2 + 8x - 5 = 0 \Rightarrow 3x^2 - 8x + 5 = 0 \Rightarrow 3x^2 - 3x - 5x + 5 = 0$
$3x(x-1) - 5(x-1) = 0 \Rightarrow (x-1)(3x-5) = 0 \Rightarrow x = 1, \frac{5}{3}$

② Make intervals and test which are increasing/decreasing based on sign of the derivative at a point on the interval:

Intervals: $(-\infty, 1), (1, \frac{5}{3}), (\frac{5}{3}, \infty)$



Testing point within $(-\infty, 1)$ $x = 0$:
$g'(0) = -3(0)^2 + 8(0) - 5 = -5$ ← negative
g is decreasing within $(-\infty, 1)$

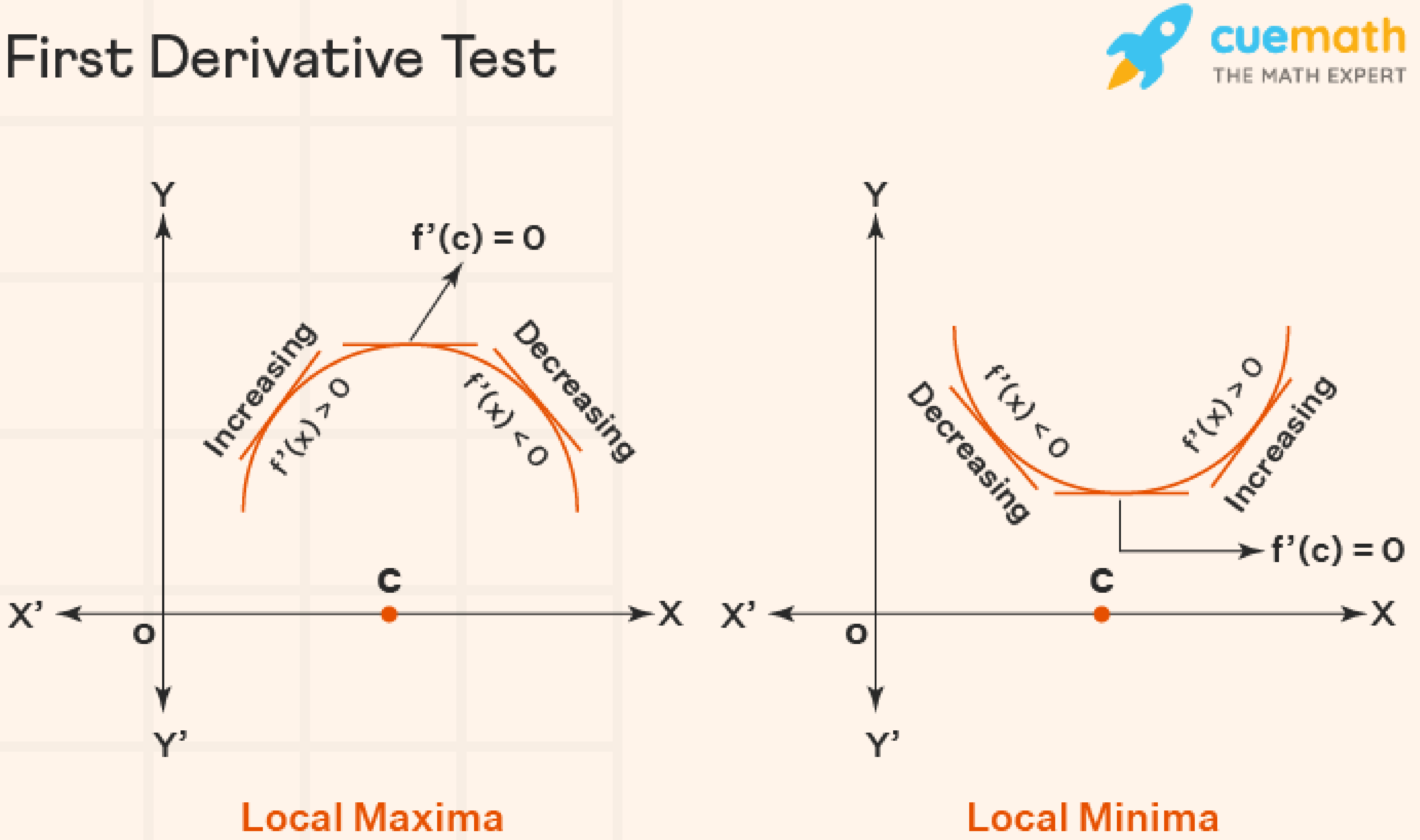
Testing point within $(1, \frac{5}{3})$ $x = \frac{3}{2}$
$g'(\frac{3}{2}) = -3(\frac{9}{4}) + 8(\frac{3}{2}) - 5 = -\frac{27}{4} + 12 - 5 = -\frac{27}{4} + \frac{28}{4} = \frac{1}{4}$ ← positive
g is increasing within $(1, \frac{5}{3})$

Testing point within $(\frac{5}{3}, \infty)$ $x = 2$
$g'(2) = -3(4) + 8(2) - 5 = -12 + 16 - 5 = 4 - 5 = -1$ ← negative
g is decreasing within $(\frac{5}{3}, \infty)$

So, g is increasing only within $(1, \frac{5}{3})$ or $1 < x < \frac{5}{3}$

5.4 First Derivative Test for Local Extrema

The goal of the first derivative test is to find local extrema (max or min). We can determine if a critical point is a max, min, or neither by looking at the sign of the derivative on either side of the critical point. Here is the visualization:



5.4 First Derivative Test for Local Extrema

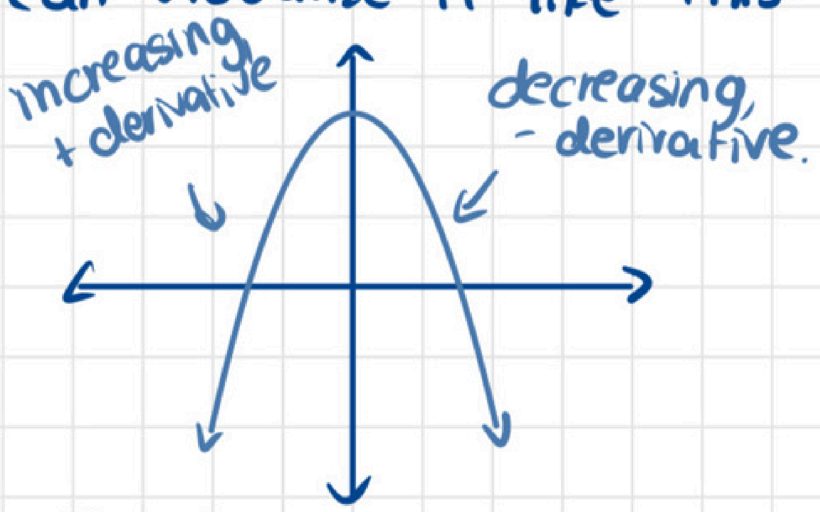
1. Find all critical points and make intervals based on those. Plot the intervals on a numberline.
2. Test a point within each interval to determine the sign of the derivative within each interval. Each interval will only have one sign. If an interval were to have more than one sign, it would have to pass through 0 which we accounted for by plotting our critical points.
3. Determine if max, min, or none:
   - If derivative goes from + to –, the critical point in the middle is a max.
   - If derivative goes from – to +, the critical point in the middle is a min.
   - If the derivative doesn't change signs, the critical point in the middle is neither a max or min.

5.4 First Derivative Test for Local Extrema

Let f be a polynomial function and let f', its **derivative**, be defined as $f'(x) = x^4(x-2)(x+3)$.

At how many points does the graph of f have a relative *maximum*?

① f has a relative maximum where the derivative goes from + to −, or f goes from increasing to decreasing. You can visualize it like this:



② For the derivative to go from + to −, it must pass through 0, so we find all points where $f'(x)=0$. This occurs when $x = 0, 2, -3$

③ Make a numberline and find where the derivative is positive and negative.

$f'(x)$: + 0 − 0 − 0 +
x: −3, 0, 2

Testing interval $(-\infty,-3)$ w/ $x=-4$
$f'(-4) = (-4)^4(-6)(-1)$ — reduce to signs since that's what we care about
$(-)^4(-)(-)$
$(+)(-)(-) = +$, increasing

Testing interval $(-3,0)$ w/ $x=-1$
$f'(-1) = (-1)^4(-3)(2)$
$(-)^4(-)(+)$
$(+)(-)(+) = -$, decreasing

Testing interval $(0,2)$ w/ $x=1$
$f'(1) = (1)^4(-1)(4)$
$(+)^4(-)(+) = -$, decreasing

Testing interval $(2,\infty)$ w/ $x=3$
$f'(3) = (3)^4(1)(4)$
$(+)^4(+)(+) = +$, increasing

④ Since $f'(x)$ goes from + to − at $x=-3$, that's where the relative max occurs. There is only one relative max.

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5.5 Candidate's Test for Global Extrema

This test refers to the "candidates" for being global maximums or minimums. Global just means overall maximum or minimums. There might be a few relative max or mins, but there's only one max that's greater than the rest or one min that's smaller than the others.

The candidates we have to evaluate are the critical points and the endpoints.

For a closed interval, we evaluate the critical points and the endpoints and compare their f(x) values to find the greatest and smallest f(x) values, giving us our min or max.

For an open interval, we evaluate only the critical points, since infinity is not an endpoint we can evaluate at.

5.5 Candidate's Test for Global Extrema

Let $g(x) = x^3 - 12x + 7$.

The absolute *maximum* value of g over the closed interval $[-4, 5]$ occurs at what x-value?

We use Candidate's Test w/ canditates being the endpoints and relative min/max s. We make a table w/ these candidates and compare them to find the greatest and smallest values.

① Find relative min/max: $g'(x) = 3x^2 - 12 = 0$ ← critical points

$x^2 - 4 = 0$ $x = \pm 2$ ←

② Make table w/ critical points and endpoints:

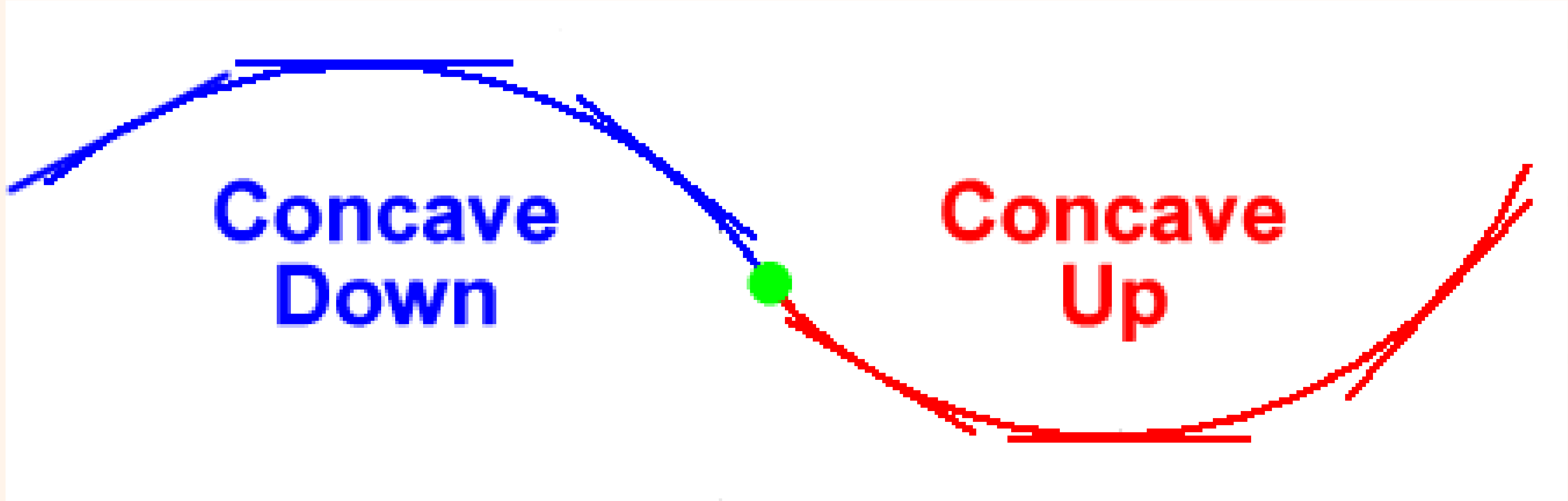
x	$g(x) = x^3 - 12x + 7$	$g(x)$
-4	$g(-4) = (-4)^3 - 12(-4) + 7 = -64 + 48 + 7$	-7
-2	$g(-2) = (-2)^3 - 12(-2) + 7 = -8 + 24 + 7$	23
2	$g(2) = (2)^3 - 12(2) + 7 = 8 - 24 + 7$	-9
5	$g(5) = (5)^3 - 12(5) + 7 = 125 - 60 + 7$	72

← absolute max, highest $g(x)$ value at $x = 5$

5.6 Determining Concavity of Functions Over Their Domain

- The graph of a function is concave up when the function’s derivative is increasing (second derivative positive). Think of a concave up function like being a cup able to hold water.
- The graph of a function is concave down when the function’s derivative is decreasing (second derivative negative). Think of a concave down function like being a hill that can’t hold water.
- Inflection points happen when the second derivative $f''(x) = 0$ and the second derivative changes sign (concavity changes).

5.6 Determining Concavity of Functions Over Their Domain



5.6 Determining Concavity of Functions Over Their Domain

Let $f(x) = -x^7 + 7x^6$.

For what values of x does the graph of f have a point of inflection?

Points of inflection occur when the second derivative changes signs (not only when $f''(x)=0$!)

$f'(x) = -7x^6 + 42x^5$

$f''(x) = -42x^5 + 210x^4 = 0$

$x^5 - 5x^4 = 0$

$x^4(x-5) = 0 \quad x = 0, 5$

$f''(x)$: − 0 − 0 +

x: 0, 5

Since $f''(x)$ only changes sign at $x=5$, the only inflection point is at $x=5$.

5.7 Second Derivative Test for Extrema

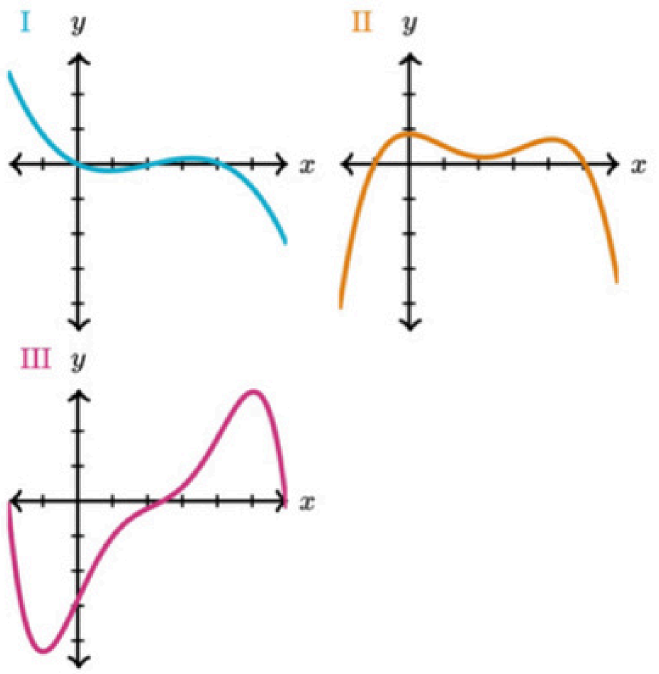
- If the second derivative of a function at a critical point is positive, the critical point is a relative min. If the second derivative of a function at a critical point is negative, the critical point is a relative max. (Only if the critical point is a min/max, not neither. In other words, the derivative must change signs at that point for this to apply.)

5.7 Second Derivative Test for Extrema

- If the second derivative of a function at a critical point is positive, the critical point is a relative min. If the second derivative of a function at a critical point is negative, the critical point is a relative max. (Only if the critical point is a min/max, not neither. In other words, the derivative must change signs at that point for this to apply.)

5.9 Connecting a Function to Its First and Second Derivatives

Let h be a twice differentiable function. One of these graphs is the graph of h, one is of h' and one is of h''.



Choose the option that matches each function with its appropriate graph.

We can use the fact that the derivative of an even polynomial is odd and vice versa.
I and III odd (ends go off to different sign infinities), II is even (ends go to same sign infinity).
So, II must be the middle $h'(x)$.

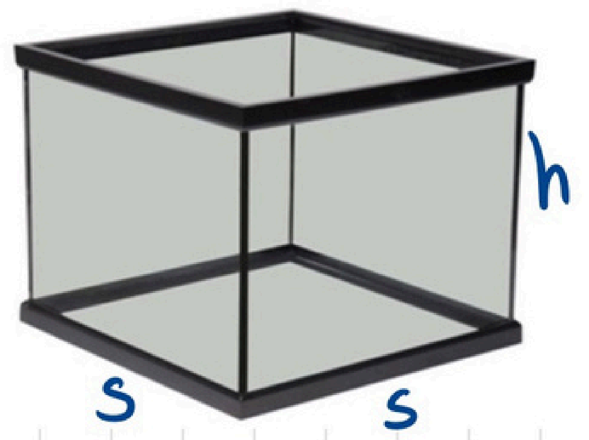
Since III's graph has slope of 0 when II's graph hits $y=0$, II is the derivative of III. Therefore, III is $h(x)$ and I is $h''(x)$.

5.10 Optimization

The goal of optimization is to represent a situation with equations, then relate those equations and find the relative max/min of the combined equation. There are far better places that explain optimization, so I'll just leave a practice problem here. Email me through dianamoya@loopsofkindness.com for help if you need!

5.10 Optimization

An open-topped glass aquarium with a square base is designed to hold 62.5 cubic feet of water. What is the minimum possible exterior surface area of the aquarium?



① Label the diagram. Since we have a square base, the length and with can both be s (they're equal!) They did not tell us if the height is the same, so let's give it it's own name, h.

② What we know:
Designed to hold 62.5 ft^3, so the volume $V = 62.5$
Volume of a rectangular prism is given by
$V = (\text{length})(\text{width})(\text{height})$
Since our length = width = s, we have
$V = (s)(s)(h) = s^2h = 62.5$
Surface area is all the surfaces added up. We have 4 sides and one bottom (no top since the the problem told us it's open-topped)
Each side has area sh and the bottom has area s^2, so all of them added up, $SA = 4sh + s^2$

$$\begin{array}{r} {}^{1\,2}\\ 62.5 \\ \times 4 \\ \hline 250.0 \end{array}$$

③ Substitute equations. Since $s^2h = 62.5$, $h = \frac{62.5}{s^2}$

We can substitute $h = \frac{62.5}{s^2}$ into the second equation:

$$SA = 4sh + s^2 = 4s\left(\frac{62.5}{s^2}\right) + s^2$$
$$= \frac{250}{s} + s^2$$

This step allows us to find the relative min of SA in the next step since we only have to worry about one variable now.

5.10 Optimization

④ Find relative min: ← find critical points

$SA' = -\frac{250}{s^2} + 2s = 0 \Rightarrow -250 + 2s^3 = 0 \Rightarrow s^3 = 125 \Rightarrow s = 5$

SA': − | +

s: ←——|——→ (5)

Since SA' goes from − to + at $s = 5$, the relative min is there.

⑤ Plug in to SA

$SA(5) = \frac{250}{5} + (5)^2 = 50 + 25 = \boxed{75 \text{ ft}^2}$